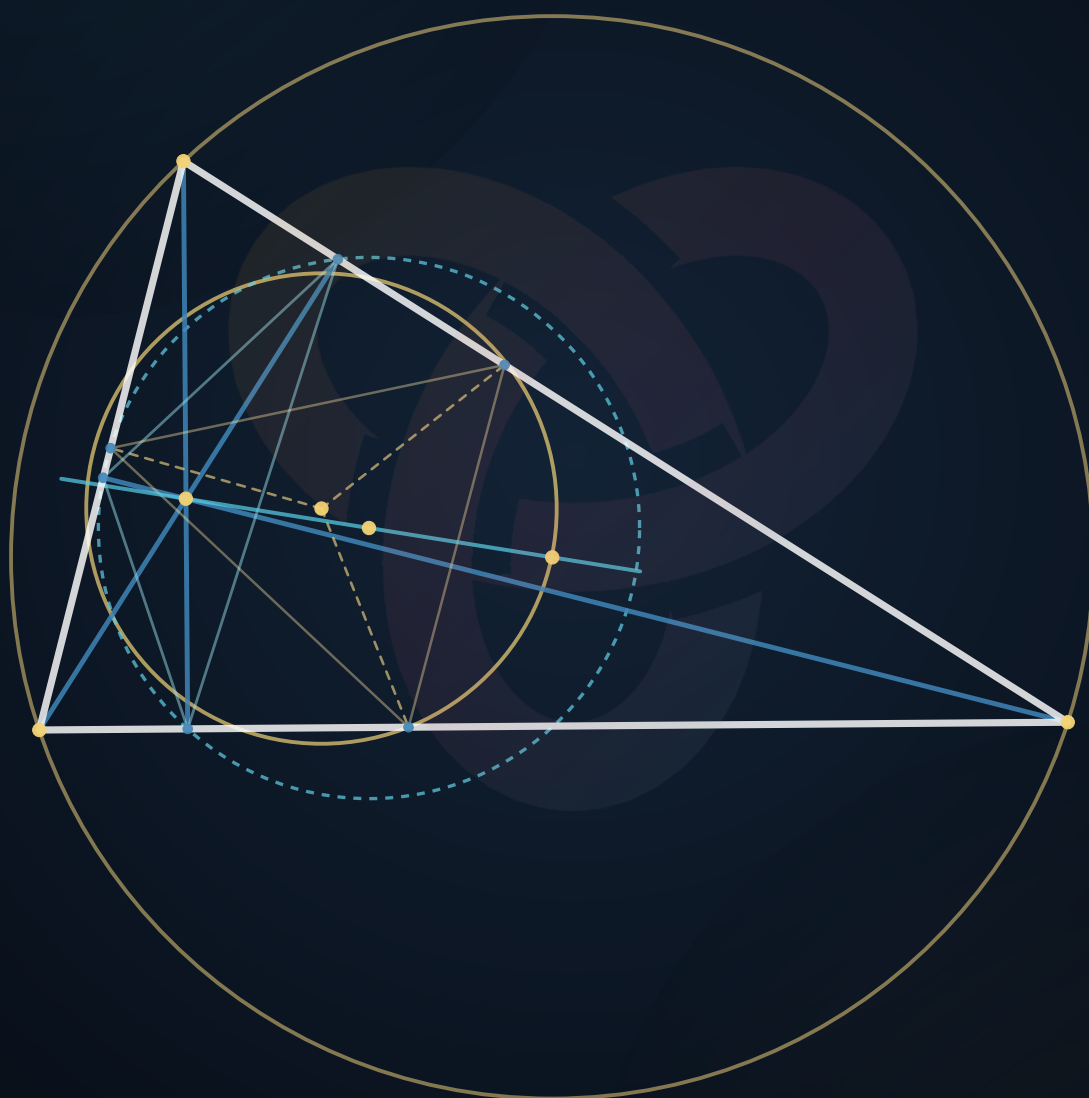


BEYOND THE INCENTER

A Collection of 100 Geometry Problems
For Mathematical Olympiads



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4 Cross Ratios

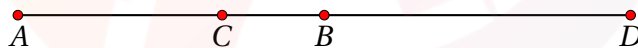
In this chapter, we will delve into projective geometry, which unlike Euclidean, views geometry in a different way.

4.1 Harmonic Points

Consider four collinear points A, B, C, D with C and D lying on and outside segment AB respectively, as shown below. We call these points harmonic if and only if C, D split AB internally and externally in the **same ratio**. In other words,

$$\frac{AC}{CB} = \frac{AD}{DB}$$

We say that C and D harmonically split segment AB , or C and D are harmonic conjugates with respect to AB . Note that $A, B; C, D$ form a **harmonic range**. We notice that if $A, B; C, D$ represent a harmonic range, then $C, D; A, B$ also form a harmonic range, and an arbitrary point P not on line AD would create a **harmonic bundle** or **pencil** $PA, PB; PC, PD$. Before we go into theory and discuss the cross ratio, consider what would happen to point D if C is the midpoint of AB . This means $\frac{AC}{CB} = 1$ and since D lies outside AB , it follows that D lies at the point of *infinity*. Hence we can also view $A, B; C, \infty$ as a harmonic range.



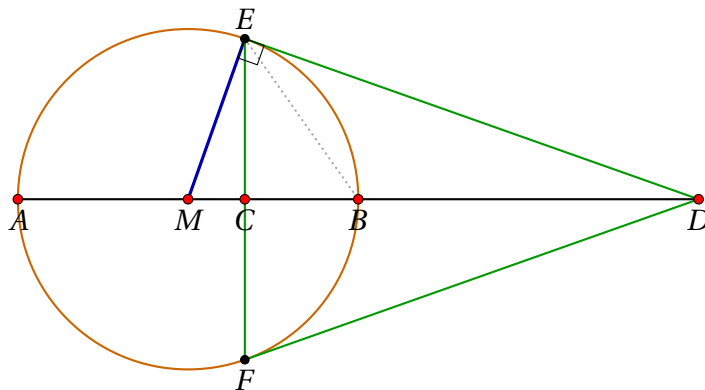
The core element of harmonic points is the *cross ratio*, which we define

$$(A, B; C, D) = \frac{\overline{CA}}{\overline{CB}} \div \frac{\overline{DA}}{\overline{DB}} \text{ representing directed segments.}$$

Thus $A, B; C, D$ are harmonic $\Leftrightarrow (A, B; C, D) = -1$

4.2 Related Theorems

While the definition of a harmonic cross ratio is seems clean and simple, there are many configurations where a trained eye could instantly spot a harmonic range.



Theorem 4.1. Let A, B, C, D be four collinear points with M as the midpoint of AB . Hence $(A, B; C, D) = -1$ if and only if:

1. $(C, D; A, B) = -1$;
2. $\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}$;
3. $AB \cdot CD = 2AD \cdot BC = 2AC \cdot DB$;
4. $CA \cdot CB = CM \cdot CD$;
5. $DA \cdot DB = DM \cdot DC$;
6. $MA^2 = MB^2 = MC \cdot MD$;
7. $\frac{BC}{BD} = \frac{MC}{MB}$;

Statement (1) is trivial as it is in the definition of a harmonic range.

Proof of (2):

Let $AB = 1$. Let $p = \frac{CA}{CB}$ and $q = \frac{DA}{DB}$. From $CA + CB = 1$ and $DA - DB = 1$, we have:

$$CA = \frac{p}{p+1}, \quad DA = \frac{q}{q-1}$$

Substituting into the LHS of (2):

$$\begin{aligned} \frac{1}{CA} + \frac{1}{DA} &= \frac{p+1}{p} + \frac{q-1}{q} \\ &= 2 + \left(\frac{1}{p} - \frac{1}{q} \right) = 2 \implies p = q \end{aligned}$$

Thus, the ratios of internal and external division are equal, which defines a harmonic range $(A, B; C, D) = -1$. ■

Note that statement (2) implies that AB is the harmonic mean of AC and AD .

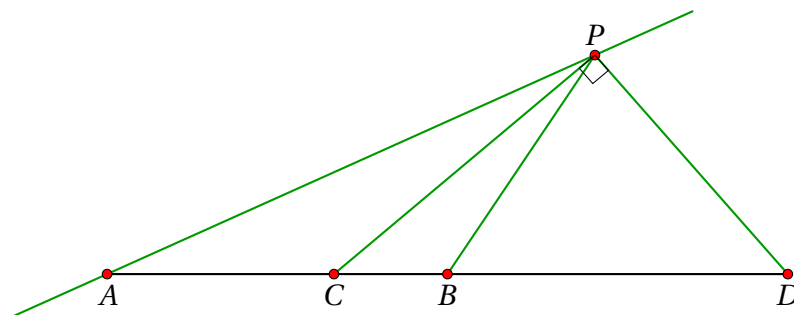
Statement (3) is a challenge left for the reader and one can use an algebraic method similar to above.

Statement (4) is fairly trivial: Note that $CA \cdot CB = CE^2 = CM \cdot CD$ as both $\triangle MFD$ and $\triangle FCD$ are right-angled.

Similarly, for statement (5), $DA \cdot DB = DE^2 = DM \cdot DC$

For statement (6), $MA^2 = MB^2 = ME^2 = MC \cdot MD$ Finally, for statement (7), note that

$$\frac{BC}{BD} = \frac{EC}{ED} = \frac{MC}{ME} = \frac{MC}{MB}$$



Theorem 4.2. *Let P be a point not on line AB . We have the following four statements:*

1. PC internally bisects $\angle APB$;
2. PD externally bisects $\angle APB$;
3. $(A, B; C, D) = -1$
4. $\angle CPD = 90^\circ$

Note that any two of the above holds true $\implies$ the rest holds true.

The proof is straightfoward and uses internal and external bisector theorem, namely $\frac{AP}{BP} = \frac{AC}{CB}$ and $\frac{AP}{BP} = \frac{AD}{BD}$, hence (1) and (2) $\implies A, B; C, D$ form a harmonic range.

Proof [(3),(4) $\implies$ (1),(2)]

Given $\frac{AC}{CB} = \frac{AD}{BD}$ and $\angle CPD = 90^\circ$. Let $\angle APC = \alpha, \angle CPB = \beta$. By Sine Rule on $\triangle APC, \triangle CPB, \triangle APD, \triangle BPD$:

$$\frac{AC}{CB} = \frac{PA \sin \alpha}{PB \sin \beta}$$

$$\frac{AD}{BD} = \frac{PA \sin(90^\circ + \alpha)}{PB \sin(90^\circ - \beta)} = \frac{PA \cos \alpha}{PB \cos \beta}$$

Equating the ratios via (3):

$$\frac{\sin \alpha}{\sin \beta} = \frac{\cos \alpha}{\cos \beta} \implies \tan \alpha = \tan \beta \implies \alpha = \beta$$

Thus PC bisects $\angle APB$ internally (1). Since $PD \perp PC$, PD is the external bisector (2). ■

It is important to note that by uniqueness of the harmonic conjugate, the remaining pairs follow from the two cases above.

Theorem 4.3. *As mentioned in previous chapters, given $A, B, k \neq 1, k > 0$, such that the set of all points P satisfying $\frac{PA}{PB} = k$ creates a circle with diameter CD , where $(A, B; C, D) = -1$, and $\frac{CA}{CB} = \frac{DA}{DB} = k$. We call this circle the Apollonius Circle and it is closely linked to harmonic cross ratios.*

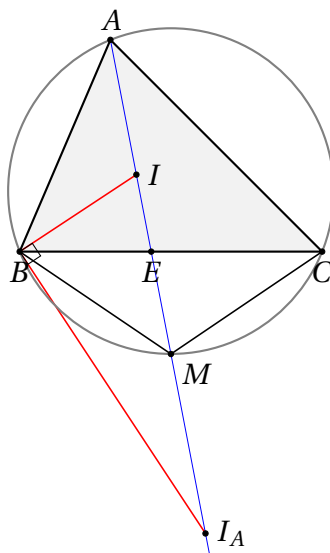
Theorem 4.4. *Construct two non-intersecting circles with centers O_1 and O_2 and let their common tangents intersect at P and Q . Note that $(O_1, O_2; P, Q) = -1$.*

Proof. Note that points P and Q are points of homothety of circles O_1 and O_2 , hence

$$\frac{PO_1}{PO_2} = \frac{R_1}{R_2} = \frac{QO_1}{QO_2}$$

and we are done. □

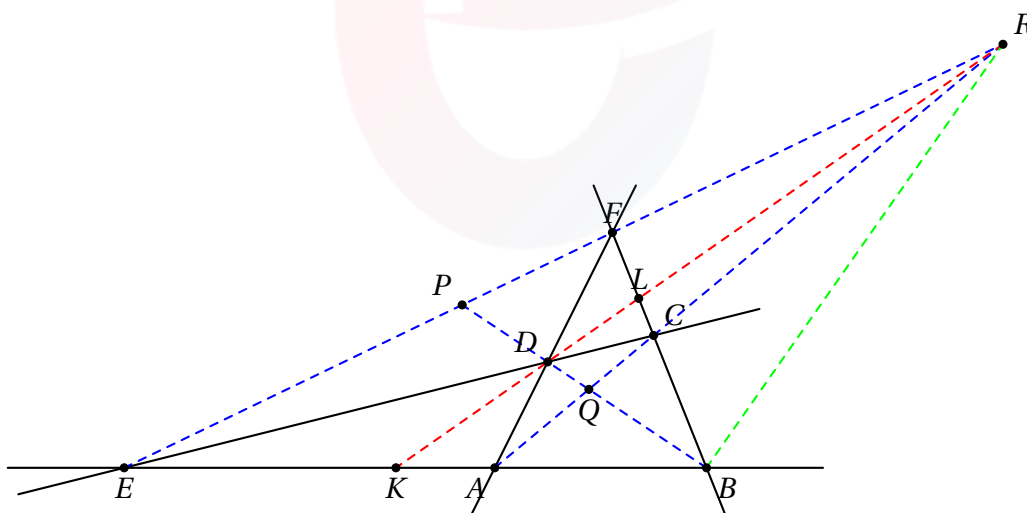
Theorem 4.5. *Denote I and I_A as the incenter and the A -excenter of $\triangle ABC$ respectively. Let $AI_A \cap BC = E$, hence $(A, E; I, I_A) = -1$. Note that $M = AI \cap \odot(ABC)$ and is the midpoint of $\widehat{BC}$ (minor arc).*



Proof. As BI and BI_A are the internal and external bisectors of $\angle ABC$ and points A, I, E, M, I_A are collinear from the Incenter-Excenter Lemma, it follows that $\angle IBI_A = 90^\circ$, hence by **Theorem 4.2**, $A, E; I, I_A$ forms a harmonic range. $\square$

4.3 Complete Quadrilaterals

A complete quadrilateral is a geometric figure formed by four coplanar lines, no three of which intersect at the same point, and the six distinct points where these lines intersect each other. An important feature is that it contains three diagonals, which are the line segments connecting the pairs of opposite vertices.



Theorem 4.6. For any complete quadrilateral, any diagonal is cut harmonically by the other two diagonals. In the diagram above,

$$(E, F; P, R) = -1, (A, C; Q, R) = -1, \text{ and } (B, D; Q, P) = -1$$

Proof. We prove this using Ceva and Menelaus. Menelaus with $\triangle BED$ and transversal AQC gives

$$\frac{BQ}{DQ} = \frac{BA}{AE} \cdot \frac{BC}{CD}$$

Now, using Ceva's theorem in $\triangle BED$ with respect to F gives

$$\begin{aligned}\frac{PB}{PD} &= \frac{BA}{AE} \cdot \frac{CF}{FD} \\ \Rightarrow \frac{QB}{QD} &= \frac{PB}{PD} \Rightarrow (B, D; Q, P) = -1\end{aligned}$$

Similarly, we can prove $(A, C; Q, R) = -1$ and $(E, F; P, R) = -1$. ■ □

Remark

Notice that if we let $RD \cap BF = L$ and $RD \cap BE = K$, then $(E, A; K, B) = -1$. Always be on the lookout for complete quadrilaterals (covered in detail next chapter) as they have many harmonic ranges and bundles which are easily *transferrable*. For example, in the diagram above, $RC ADEF$ is also a complete quadrilateral with diagonals FC , RD , and EA concurrent at B , which yields a series of new harmonic ranges.

Theorem 4.7. Referring to the diagram above, if $AC \parallel EF$, then P, Q are the respective midpoints of EF, AC , and $(B, D; Q, P) = -1$

Proof. We can let $R = AC \cap EF = \infty$, hence $(E, F; P, \infty) = -1$, hence P, Q are the respective midpoints and etc. as mentioned in Section 4.1. However, some people may feel uncomfortable using points at infinity, and this is clearly a special case of a complete quadrilateral, so should be more straightforward to prove using standard methods. It follows from Ceva's theorem that $\frac{FP}{PE} = \frac{FC}{CB} \cdot \frac{BA}{AE} = 1$ (as $\frac{FC}{CB}$ is the reciprocal of $\frac{BA}{AE}$) □

Remark

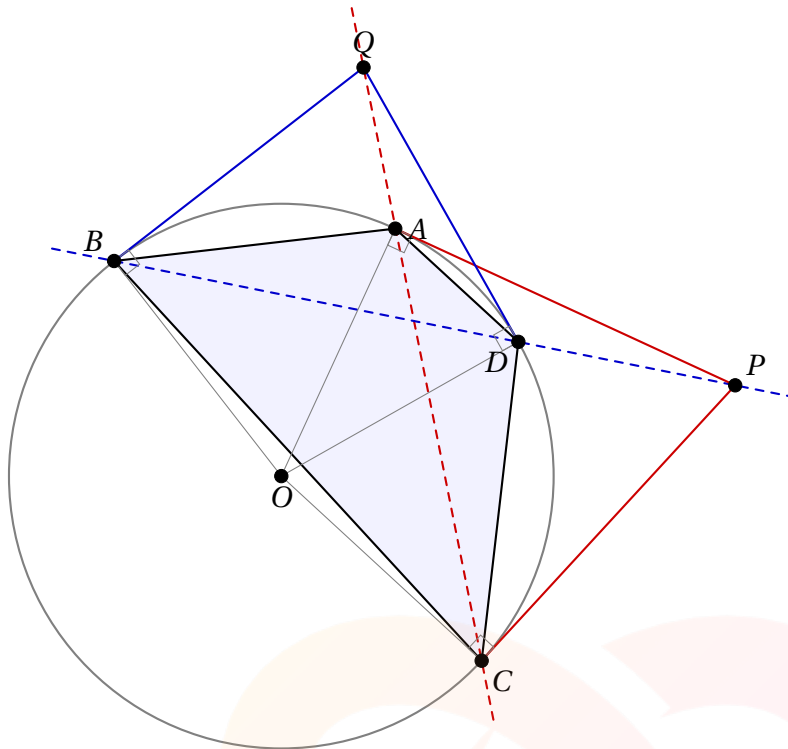
Another way of viewing this is to suppose we have AD, BE, CF concurrent, which could be a named center such as I, H , or any point in general. We should instantly spot the harmonic range $(B, C; D, G) = -1$ where $G = FE \cap BC$.

4.4 Harmonic Quadrilaterals

Theorem 4.8. Let $ABCD$ is a cyclic quadrilateral. Hence BD concurs with the tangents at A and C

$$\begin{aligned}\Leftrightarrow AD \cdot BC &= AB \cdot CD \left(= \frac{1}{2} AC \cdot BD \right) \text{ by Ptolemy's identity} \\ \Leftrightarrow AC &\text{ concurs with the tangents at } B \text{ and } D.\end{aligned}$$

Note that any cyclic quadrilateral with these properties are harmonic.



Proof. Let BD intersect tangents at A and C at P_1 and P_2 respectively. (Creation of a phantom point) Hence $\triangle P_1AD \sim \triangle P_1BA \Rightarrow \frac{P_1D}{P_1A} = \frac{P_1A}{P_1B} = \frac{AD}{BA}$

$$\Rightarrow \frac{P_1D}{P_1B} = \frac{P_1D}{P_1A} \cdot \frac{P_1A}{P_1B} = \left(\frac{AD}{BA} \right)^2 \quad (\text{Removes } P_1)$$

Similarly, $\frac{P_2D}{P_2B} = \left(\frac{CD}{BC} \right)^2$.

$\therefore BD$ concurs with tangents at A and $C \Leftrightarrow P_1, P_2$ coincide.

$$\Leftrightarrow \frac{AD}{BA} = \frac{CD}{BC}$$

$$\Leftrightarrow AD \cdot BC = AB \cdot CD$$

This is the same case for AC and tangents at B, D . □

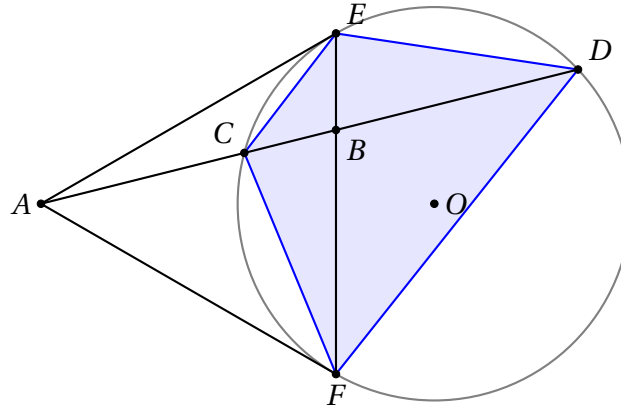
Remark

Notice that we can 'switch' sides to find new concurrency if the cyclic quadrilateral is harmonic. We will cover harmonic bundles next, but an interesting mini-theorem is as follows:

Given $ABCD$ is a cyclic quadrilateral, and point P lies on any other point on the circumcircle, we get

$$ABCD \text{ is a harmonic quadrilateral} \Leftrightarrow PA, PC; PB, PD \text{ is harmonic.}$$

Theorem 4.9. *Generalisation of theorem 4.1: Let AE, AF be tangent to $\odot O$. Let a line through A meet $\odot O$ again at C, D , and meet EF at B . Then $(A, B; C, D) = -1$.*

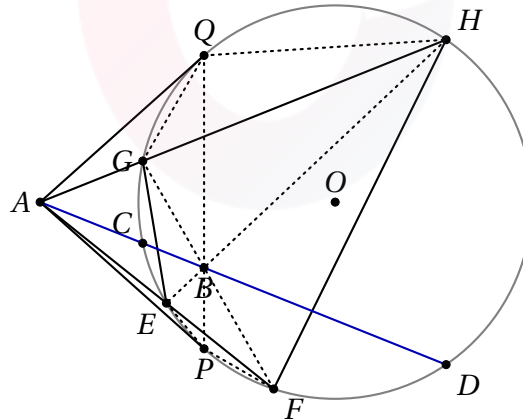


Proof. Note that

$$\begin{aligned} \frac{BC}{BD} = \frac{AC}{AD} &\Leftrightarrow \frac{[\triangle CEF]}{[\triangle DEF]} = \frac{AC}{AD} \text{ (Removes } B) \\ &\Leftrightarrow \frac{CE \cdot CF}{DE \cdot DF} = \frac{AC}{AD} \\ &\Leftrightarrow \frac{AC}{AE} \cdot \frac{AF}{AD} = \frac{AC}{AD} \end{aligned}$$

and we are done. $\square$

Theorem 4.10. *Let $EFGH$ be a cyclic quadrilateral, $FE \cap GH = A$, and $EG \cap HF = B$, and let AB meet $\odot O$ at points C and D , hence $(A, B; C, D) = -1$.*

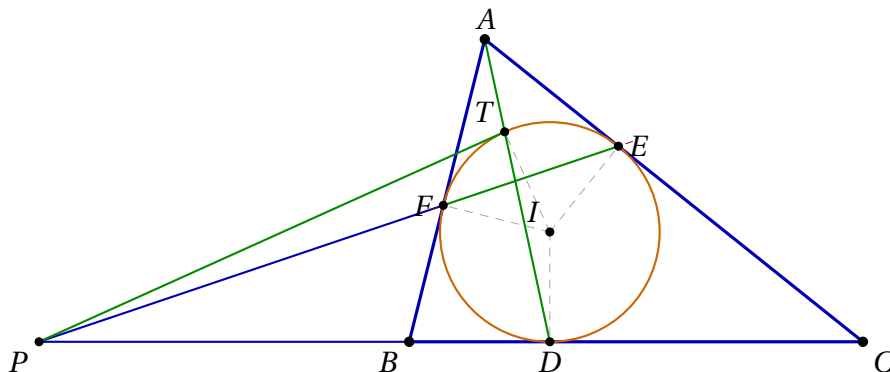


Proof. Construct tangents AP, AQ . Now, from *theorem 4.9*, it suffices to prove PQ passes through B .

$$\begin{aligned} &\Leftrightarrow PQ, EH, FG \text{ are concurrent.} \\ &\Leftrightarrow \frac{EP}{PF} \cdot \frac{FH}{HQ} \cdot \frac{QG}{GE} = 1 \text{ (Ceva's theorem)} \\ &\Leftrightarrow \frac{AE}{AP} \cdot \frac{AH}{AQ} \cdot \frac{AQ}{AH} = 1 \end{aligned}$$

and we are done. $\square$

Theorem 4.11. Let D, E, F be the points of tangency of the incircle of $\triangle ABC$ with its sides BC, AC, AB respectively. Let $FE \cap BC = P$, thus $(B, C; D, P) = -1$. Also, let $AD \cap \odot I = T$, hence PT is tangent to $\odot I$. This is the same case for the excircles of $\triangle ABC$.



Proof. Line DT and tangents at E, F concur at A .

$\Rightarrow TFDE$ is a harmonic quadrilateral.

$\Rightarrow$ Line EF and tangents at D, T are concurrent.

□

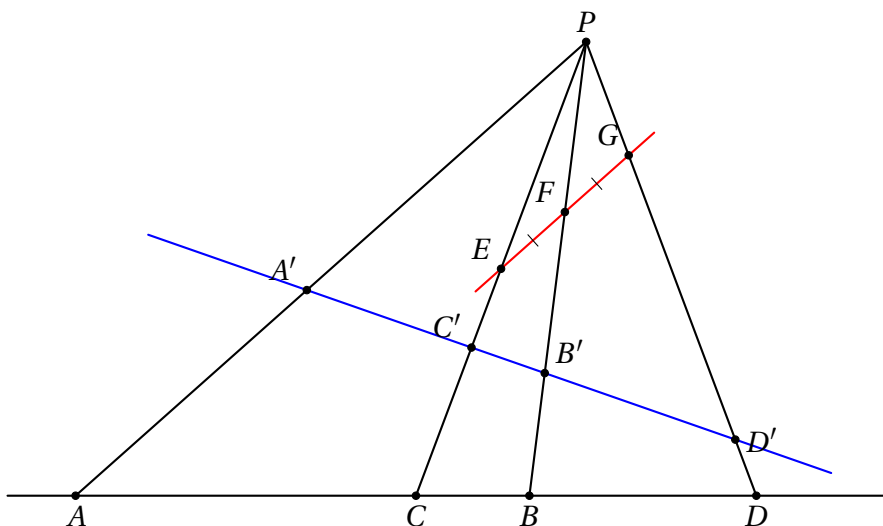
4.5 Harmonic Bundles

We say that a bundle (or pencil) of four lines (PA, PC, PB, PD) passing through a common point P is a *harmonic bundle* of lines, if there exists a line intersecting all four lines of the bundle at points A, C, B, D , such that C, D are harmonic conjugates of A, B (i.e. $P(A, B; C, D) = -1$).

Theorem 4.12. Given collinear points A, B, C, D and a point P not lying on the same line, we have

$$(A, B; C, D) = -1 \Leftrightarrow \frac{\sin \angle APC}{\sin \angle CPB} = \frac{\sin \angle APD}{\sin \angle DPB} \quad (*)$$

$\Leftrightarrow$ Any line parallel to one of PA, PB, PC, PD is divided by the remaining three lines into two segments of equal lengths. (**)



Proof. Proof of (*): For segments AC and CB :

$$\frac{AC}{CB} = \frac{[\triangle PAC]}{[\triangle PCB]} = \frac{\frac{1}{2}PA \cdot PC \sin \angle APC}{\frac{1}{2}PC \cdot PB \sin \angle CPB} = \frac{PA \sin \angle APC}{PB \sin \angle CPB}$$

For segments AD and DB :

$$\frac{AD}{DB} = \frac{[\triangle PAD]}{[\triangle PDB]} = \frac{\frac{1}{2}PA \cdot PD \sin \angle APD}{\frac{1}{2}PD \cdot PB \sin \angle DPB} = \frac{PA \sin \angle APD}{PB \sin \angle DPB}$$

Since $(A, B; C, D) = -1$, we have the following equal ratios: $\frac{AC}{CB} = \frac{AD}{DB}$.

$$\begin{aligned} \therefore \frac{PA \sin \angle APC}{PB \sin \angle CPB} &= \frac{PA \sin \angle APD}{PB \sin \angle DPB} \\ \therefore \frac{\sin \angle APC}{\sin \angle CPB} &= \frac{\sin \angle APD}{\sin \angle DPB} \end{aligned}$$

Proof of (**): Note that

$$\begin{aligned} \frac{EF}{FG} &= \frac{PE \sin \angle EPF}{PG \sin \angle FPG} = \frac{\sin \angle PGE}{\sin \angle PEG} \cdot \frac{\sin \angle CPB}{\sin \angle BPD} \\ &= \frac{\sin \angle APD}{\sin \angle APC} \cdot \frac{\sin \angle CPB}{\sin \angle BPD}, \text{ hence (**) holds true.} \end{aligned}$$

□

Remark

The proof of (*) utilises a common technique called the *Angle Division Theorem* for internal angles, and is worth remembering. Note that **the four points formed by intersection of the harmonic bundle and a straight line are harmonic**. This is because it depends on the angle relationship at the apex, rather than side lengths.

Key Takeaway

To prove harmonic ranges/bundles, we have the following strategies in a nutshell:

1. Use the definition of a harmonic range/bundle based on cross ratios.
2. Use common harmonic configurations/theorems
3. Take advantage of the interchangeability between harmonic ranges, parallel lines implying midpoints, harmonic quadrilaterals and harmonic bundles (lines). If four points are harmonic, another seemingly irrelevant line could also be harmonic due to the interchangeability of sides, for instance in the remark above.

4.6 Perspectivity and Projection

Perspectivity from a Point

Imagine a light source at a central point O . If we have four points on a line ℓ_1 and we cast their shadows from O onto a second line ℓ_2 , we create a new set of points. We say the two sets of points are *in perspective* from O .

Because projection preserves the cross-ratio, the cross-ratio of the original points equals the cross-ratio of the projected points. We denote this using an equals sign with the center of perspectivity written above it:

$$(A, B; C, D) \stackrel{O}{=} (A', B'; C', D')$$

Projecting from a Circle to a Line

Instead of projecting from one line to another, we can project four points on a circle onto a line. Let the points on the circle be X_1, X_2, X_3, X_4 . We pick a center of perspectivity O that is *also on the circle*.

To project these points onto a target line ℓ :

- We draw rays OX_1, OX_2, OX_3, OX_4 .
- We mark where each ray intersects the target line ℓ .

There are two special edge cases you must know to read projective geometry proofs:

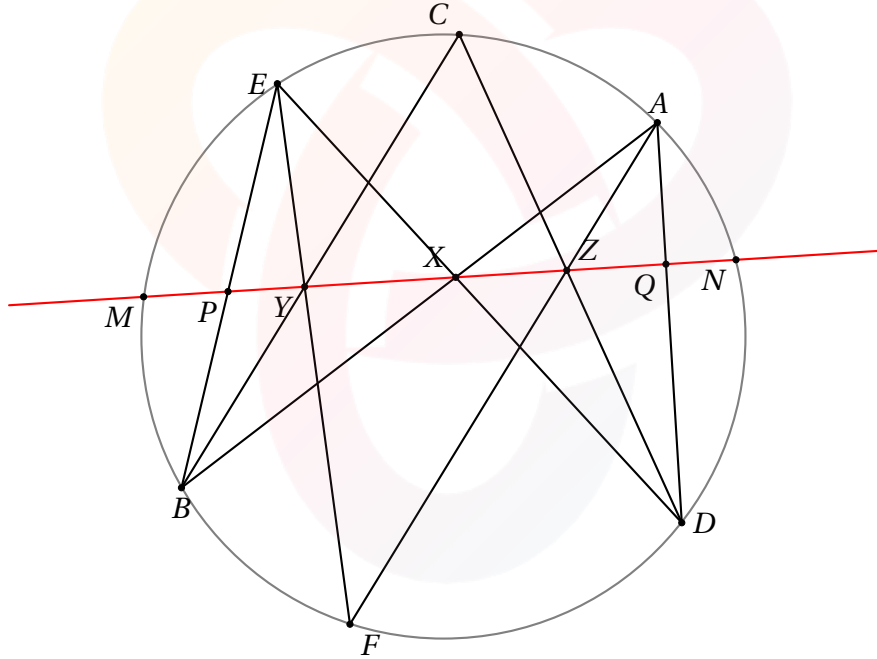
1. **Projecting a point onto itself** ($O \rightarrow O$): What is the "ray" connecting the point O to the point O ? In the limit, a secant line connecting two infinitely close points on a circle becomes the tangent line. Therefore, the ray OO is defined as the **tangent to the circle at O** .
2. **Parallel Rays and Infinity**: If one of the projecting rays happens to be perfectly parallel to the target line ℓ , they will never intersect on the Euclidean plane. In projective geometry, parallel lines meet at a "point at infinity," denoted as P_∞ .

4.7 Further Reading – Pascal's Theorems

The following theorems yield powerful results, yet they rarely appear in Olympiad problems. I will provide two proofs to these theorems, one by Trigonometric Ceva, and one by Projective geometry, which is the core of this chapter. Whilst these theorems are not strictly projective in their classical definitions, I have decided to include them in this chapter to highlight the power of projective geometry, as they demonstrate how a projective lens allows us to view Euclidean geometry differently. My hope is that the next time you quote these theorems in a proof, a quiet corner of your mind would whisper, "projective".

Pascal's Theorem

Theorem 4.13 (Pascal). *Consider cyclic hexagon $ABCDEF$ inscribed in Γ , and let $X = AB \cap DE$, $Y = BC \cap EF$, $Z = CD \cap FA$, hence X, Y, Z are collinear. Conversely, let $ABCDE$ be 5 concyclic points which lie on Γ , and let F be a point on the plane such that X, Y, Z are collinear, hence F lies on Γ .*



Proof 1 (Trig Ceva). X, Y, Z are collinear if and only if the vertical angles at X match:

$$\begin{aligned} \angle EXY = \angle DXZ &\iff \angle BXY = \angle AXZ \\ &\iff \frac{\sin \angle EXY}{\sin \angle BXY} = \frac{\sin \angle DXZ}{\sin \angle AXZ} \end{aligned}$$

By applying the Trigonometric form of Ceva's Theorem (which removes the dependency on the intersections X, Y, Z), this equality essentially becomes:

$$\begin{aligned} \frac{\sin \angle CBE}{\sin \angle CBA} \cdot \frac{\sin \angle DEF}{\sin \angle BEF} &= \frac{\sin \angle EDC}{\sin \angle ADC} \cdot \frac{\sin \angle DAF}{\sin \angle BAF} \\ \because \angle CBE &= \angle EDC, \quad \angle CBA = \angle CDA, \quad \angle BEF = \angle BAF, \quad \angle DEF = \angle DAF \end{aligned}$$

Substituting these into our expansion, we reach tautology. □ □

Proof 2 (Projective). Projecting through F onto Γ , we get:

$$(M, N; Y, Z) = (M, N; E, A)$$

Now, projecting through C onto Γ , we get:

$$(M, N; Y, Z) = (M, N; B, D)$$

Projecting through D onto MN , we get:

$$(M, N; E, A) = (M, N; ED \cap MN, Q)$$

Now, projecting through A onto MN , we get:

$$(M, N; B, D) = (M, N; AB \cap MN, Q)$$

$$\therefore AB \cap MN = ED \cap MN = X$$

$$\therefore X, Y, Z \text{ are collinear.}$$

□

We will now prove the converse of Pascal's Theorem using the method of indiscernibility, a powerful technique in Olympiad geometry.

Proof. Let AF intersect Γ at a point F' , hence we can apply Pascal's Theorem in cyclic hexagon $ABCDEF'$.

$$\implies Z, X, EF' \cap BC \text{ are concurrent.}$$

$$\implies EF', BC, XZ \text{ are concurrent, hence at } Y \text{ where } Y = BC \cap XZ.$$

$$\implies E, F', Y \text{ are collinear, so } F' \text{ lies on } EY, \text{ hence } F' = AZ \cap EY.$$

$$\implies F' \text{ coincides with } F, \text{ meaning } F \text{ lies on } \Gamma.$$

□

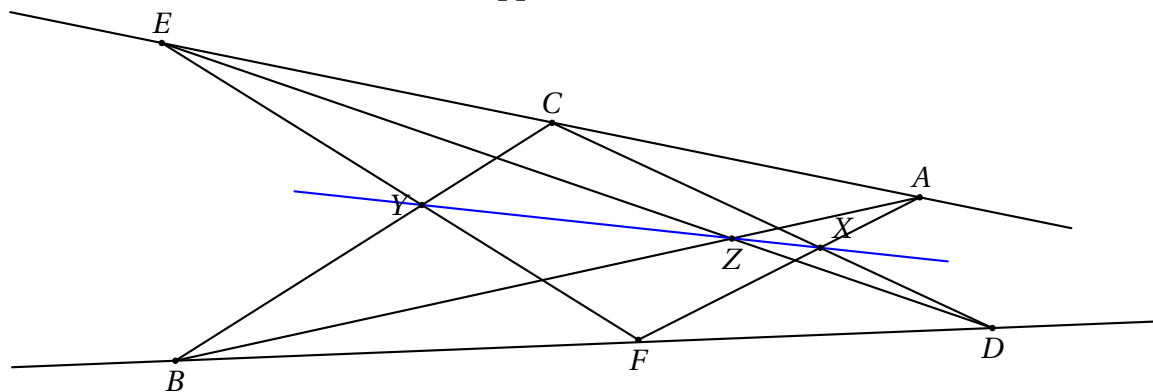
Theorem 4.14 (Pascal's Degenerate Case). *Cyclic hexagon $ABCDEF$ degenerates into cyclic quadrilateral $ABBCDD$ in Γ . Then, let $X = AB \cap CD$, $Y = BB \cap DD$, $Z = BC \cap AD$, hence points X, Y, Z are collinear. (Note that by selecting two points infinitesimally close to each other such that they coincide at B , we obtain BB which is tangent to Γ at point B).*

Remark

This configuration is very common in Geometry problems although it requires us to have two tangents to Γ at points B, D .

Theorem 4.15 (Pappus). *Let A, C, E and B, F, D be two sets of three points on a straight line. Let $X = AB \cap DE$, $Y = BC \cap EF$, $Z = CD \cap FA$, then X, Y, Z are collinear.*

Pappus' Theorem



Proof.

X, Y, Z are collinear.

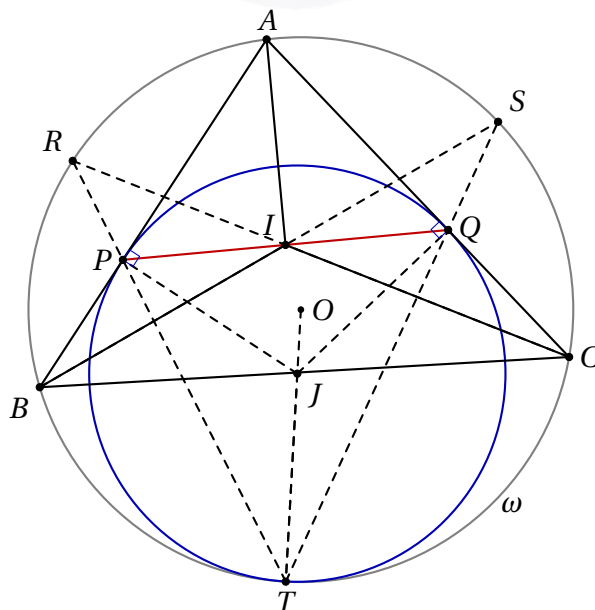
$\Leftrightarrow AB, DE, YZ$ are concurrent.

Let $X_1 = AB \cap YZ$, $X_2 = DE \cap YZ$. Hence it suffices to prove X_1, X_2 overlap.

$$\begin{aligned}
 &\Leftrightarrow \frac{YX_1}{X_1Z} \cdot \frac{X_2Z}{YX_2} = 1 \\
 &\Leftrightarrow \frac{S_{\Delta YAB}}{S_{\Delta ZAB}} \cdot \frac{S_{\Delta ZDE}}{S_{\Delta YDE}} = 1 \\
 &\Leftrightarrow \frac{S_{\Delta YAB}}{S_{\Delta YEB}} \cdot \frac{S_{\Delta YEB}}{S_{\Delta YDE}} \cdot \frac{S_{\Delta ZDE}}{S_{\Delta ZAD}} \cdot \frac{S_{\Delta ZAD}}{S_{\Delta ZAB}} = 1 \\
 &\Leftrightarrow \frac{AC}{CE} \cdot \frac{BF}{FD} \cdot \frac{EC}{CA} \cdot \frac{DF}{FB} = 1 \text{ which is trivial.}
 \end{aligned}$$

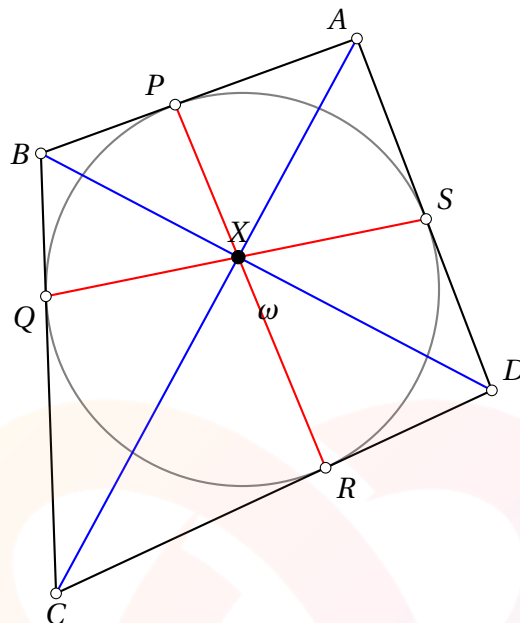
□

Theorem 4.16 (Mannheim's Theorem). *We have $\triangle ABC$ with incenter I and circumcircle ω . Extend CI and BI such that they intersect ω again at points R and S , respectively. Construct another circle tangent to AB , AC , and ω at P , Q , and T , respectively, hence P, I, Q are collinear.*



Try proving this yourself using *Pascal's Theorem*!

Theorem 4.17. Consider a quadrilateral $ABCD$ which inscribes a circle ω , and let ω be tangent to AB, BC, CD, DA at P, Q, R, S respectively, and diagonals $BD \cap AC = X$, then X lies on PR and SQ .



Proof. We first prove RP, SQ, AC are concurrent. Let $X = RP \cap AC, X' = SQ \cap AC$, hence

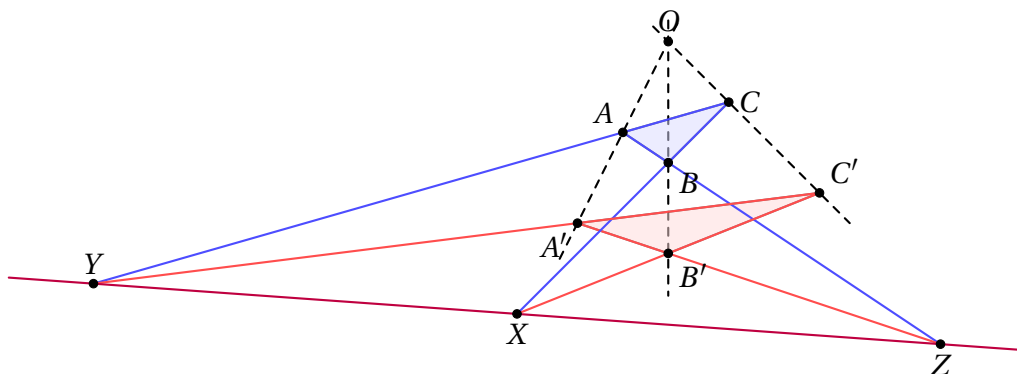
$$\frac{AX}{XC} = \frac{S_{\triangle ARP}}{S_{\triangle CRP}} = \frac{\frac{1}{2}PA \cdot PR \cdot \sin \angle APR}{\frac{1}{2}RC \cdot RP \cdot \sin \angle CRP} = \frac{PA}{RC}$$

$$\text{Similarly, } \frac{AX'}{X'C} = \frac{SA}{QC}$$

$$\therefore \frac{AX}{XC} = \frac{AX'}{X'C} \Rightarrow X \text{ coincides with } X'.$$

Similarly, we can prove RP, SQ, BD are concurrent. □

Theorem 4.18 (Desargues' Theorem). Consider two triangles $\triangle ABC$ and $\triangle A'B'C'$ and let $X = BC \cap B'C', Y = CA \cap C'A',$ and $Z = AB \cap A'B'.$ If AA', BB', CC' concur at a point O , then X, Y, Z are collinear, and conversely, if X, Y, Z are collinear, then AA', BB', CC' either concur at a point O or are parallel.



Proof. First, we prove the forwards direction (*) (AA', BB', CC' intersect at O , hence X, Y, Z are collinear).

$$\Leftrightarrow \prod \frac{AZ}{ZB} = \frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1 \text{ (from Menelaus in } \triangle ABC \text{ with transversal } XYZ)$$

$$\Leftrightarrow \prod \left(\frac{AA'}{A'O} \cdot \frac{OB'}{B'B} = 1 \right) \text{ (from Menelaus in } \triangle OAB \text{ with transversal } A'B'Z)$$

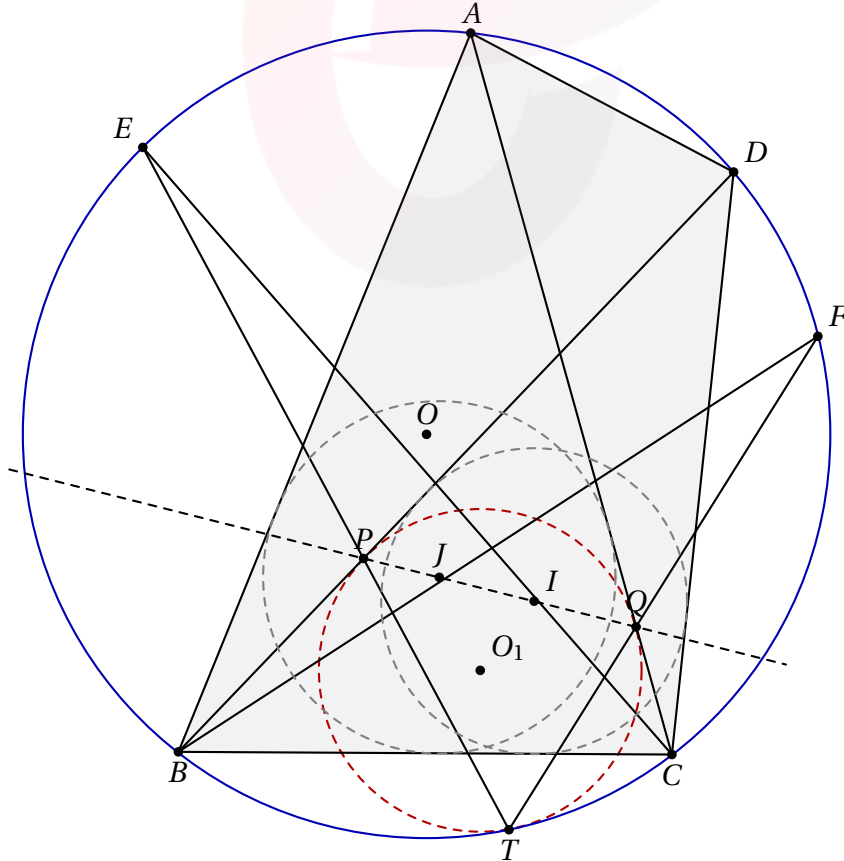
which is clearly true, hence X, Y, Z are collinear. Now, we prove the converse. If X, Y, Z are collinear and AA', BB', CC' are not all parallel, WLOG let $BB' \cap CC' = O$.

Consider $\triangle ZBB'$ and $\triangle YCC'$. The lines joining their corresponding vertices ($ZY, BC, B'C'$) concur at X . By the forward direction (*), the intersections of their corresponding sides must be collinear:

$$\begin{aligned} BB' \cap CC' &= O \\ ZB' \cap YC' &= A'B' \cap A'C' = A' \\ ZB \cap YC &= AB \cap AC = A \\ \Rightarrow O, A', A &\text{ are collinear.} \end{aligned}$$

□

Theorem 4.19 (Sawayama's Lemma). *Convex cyclic quadrilateral $ABCD$ has circumcircle Γ . A circle tangent to Γ intersects segment BC and intersects BD, AC at P, Q respectively. Thus the incenters of $\triangle ABC$ and $\triangle DBC$ both lie on line PQ .*



Proof. Let I denote the incenter of $\triangle DBC$, so we first prove I, P, Q are collinear.

Extend TP, TQ such that they intersect $\odot(ABC)$ at E, F respectively.

Hence, E is the midpoint of $\widehat{BD}$ and F is the midpoint of $\widehat{AC}$

$$\Rightarrow CE \text{ bisects } \angle BCD.$$

Let $I' = CE \cap PQ$, hence it suffices to prove I' is the incenter of $\triangle DBC$, or $EB = EI'$, or $\angle EBI' = \angle EI'B$, or $\angle DBI' = \angle CBI'$, which results in I' being the incenter of $\triangle DBC$. Note that

$$EB = EI' \text{ (from the Incenter-Excenter Lemma)} \Leftrightarrow EI'^2 = EB^2 = EP \cdot ET$$

$$\Leftrightarrow \triangle EPI' \sim \triangle EI'T \Leftrightarrow \angle EPI' = \angle EI'T$$

$$\Leftrightarrow \angle TPQ = \angle TI'C \Leftrightarrow \angle TQC = \angle TI'C \Rightarrow I', T, C, Q \text{ are concyclic.}$$

This is clearly true by some straightforward angle chasing:

$$\angle TQI' = \angle TQP = \angle PTF = \angle ETF = \angle TCE = \angle TCI'$$

and we are done. □

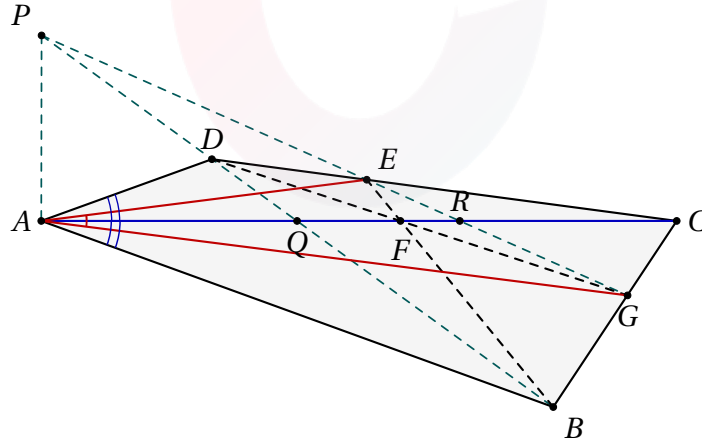
4.8 Related Problems with Solutions

Problem 50

In convex quadrilateral $ABCD$, diagonal AC bisects $\angle BAD$. Let E be a point on segment CD , and let $F = BE \cap AC$, $G = DF \cap BC$.

Prove that $\angle GAC = \angle EAC$.

(China 1999)



Solution: Let $P = BD \cap GE$, $Q = AC \cap BD$, and $R = AC \cap EG$. Notice that $CGBFDE$ is a complete quadrilateral.

$$\Rightarrow (B, D; Q, P) = -1 \text{ (by Theorem 4.6), } AQ \text{ bisects } \angle BAD.$$

$$\Rightarrow \angle QAP = 90^\circ \text{ (by Theorem 4.2 pt. 4)}$$

$$\Rightarrow \angle RAP = 90^\circ, (G, E; R, P) = -1.$$

$$\Rightarrow AR \text{ bisects } \angle GAE.$$

■

Remark

We have to be very sensitive switching from bundle $A(B, D; Q, P) = -1$ to $A(G, E; R, P) = -1$ since R is an extension of AQ . Let us consider when $GE \parallel BD$.

$\Rightarrow Q, R$ are midpoints of BD and GE respectively.

$\Rightarrow AQ \perp BD$

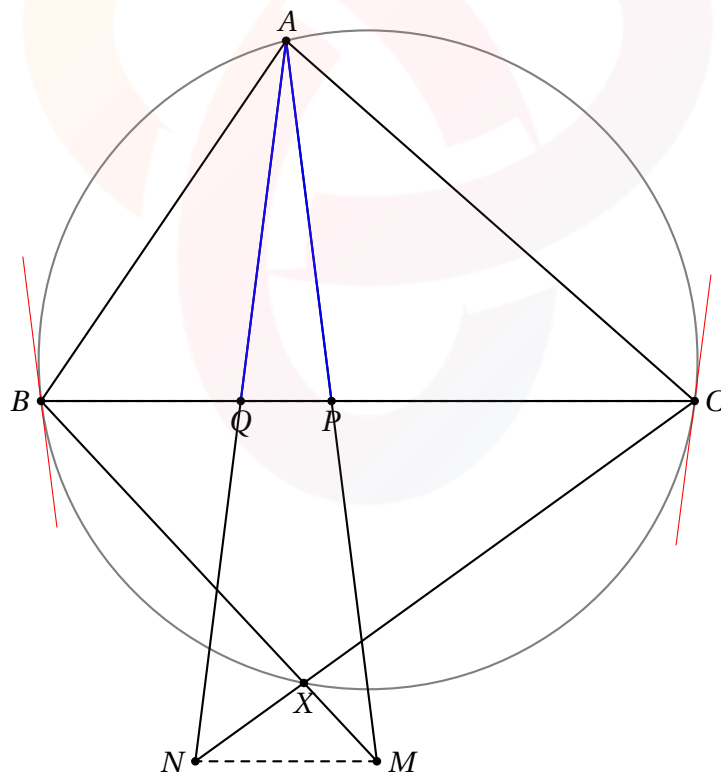
$\Rightarrow AR \perp GE$

$\Rightarrow AR$ bisects $\angle GAE$

Problem 51

Points P and Q lie on side BC of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ , respectively, such that P is the midpoint of AM , and Q is the midpoint of AN . Prove that lines BM and CN intersect on the circumcircle of $\triangle ABC$.

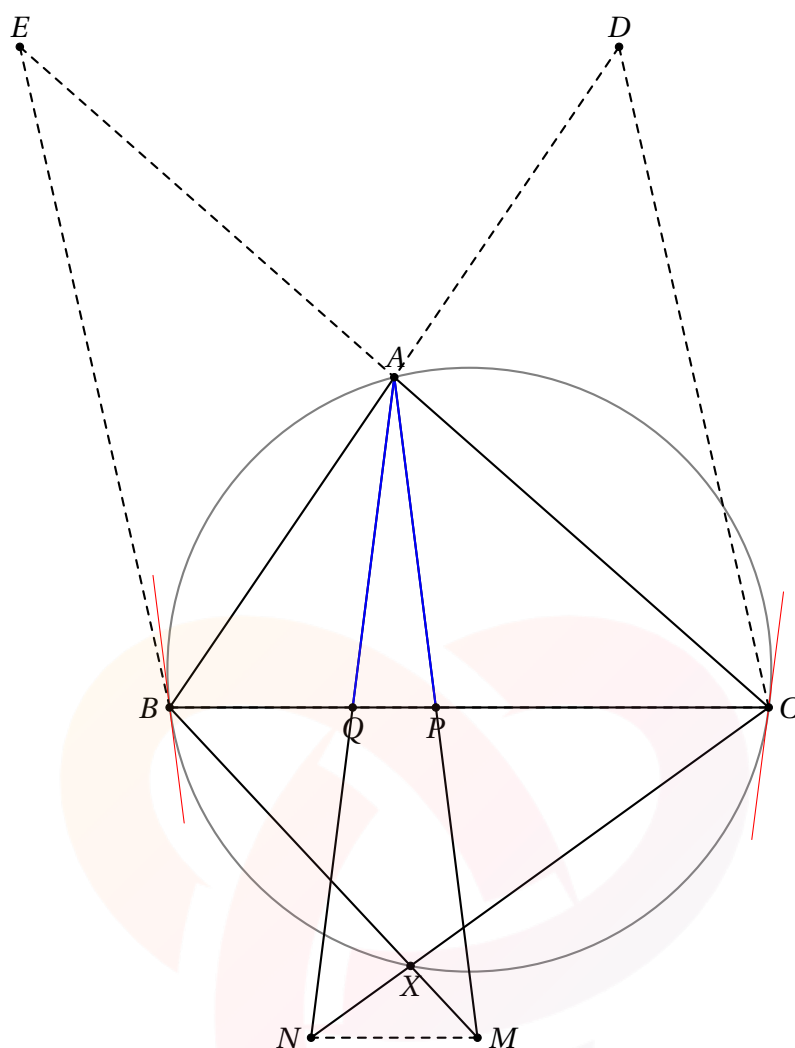
(IMO 2014 P4)



Solution 1 (Harmonic Bundles/Quadrilaterals): Let D_1 be the second intersection of NC with the circumcircle of $\triangle ABC$, and D_2 the second intersection of MB with the circumcircle of $\triangle ABC$. By inscribed angles, the tangent at C is parallel to AN . Let P_∞ denote the point at infinity along line AN . Note that

$$(A, D_1; B, C) \stackrel{C}{=} (A, N; Q, P_\infty) = -1.$$

So, ABD_1C is harmonic. Similarly, we can find ABD_2C is harmonic. Therefore, $D_1 = D_2$, hence BM and CN intersect on the circumcircle. ■



Solution 2 (Similar Triangles): The angle conditions imply that $\triangle ABC \sim \triangle QAC$. Consider reflecting point B in point A to obtain D , hence $BD = 2BA$ and $MN = 2PQ$, so it follows that $\triangle CAN \sim \triangle CBD$. We can construct a similar point E by reflecting C in A , hence $\triangle BAM \sim \triangle BCE$. Hence for $ABXC$ to be cyclic, it suffices to show $\angle ABM + \angle ACN = 180^\circ$. Note that $\angle ABM + \angle ACN = \angle CBE + \angle BCD = 180^\circ$ as $BE \parallel CD$ (easily proved by congruent triangles within $BEDC$, proving it is a parallelogram), and we are done. ■

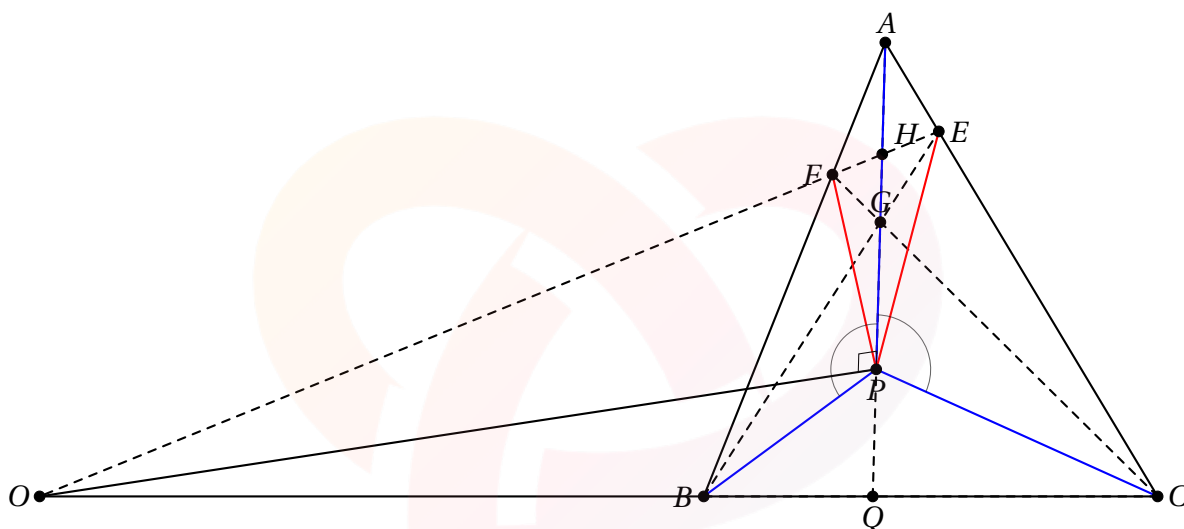
Remark

One can easily spot $\triangle ABC \sim \triangle QAC$ from the given angle conditions, but to utilise the midline PQ in such a way is rather unexpected. When we notice similar triangles, and midlines (representing double), always consider doubling the other corresponding side, as we get a parallelogram as mentioned in **Chapter 1**. However, the harmonic bundle solution is clearly favoured, using the angle condition to spot parallel lines (tangent at C and AN , etc.), and projecting a quadrilateral onto a line. This is not too difficult to think of since once we see parallel lines and midlines, and this question is in the Harmonic Cross Ratio chapter, so we should consider projection at infinity P_∞ . Please refer to **4.1 Harmonic Points** for more information on this.

Problem 52

Let P be a point inside $\triangle ABC$ such that $\angle BPA = \angle CPA$. A point G lies on the segment AP . Let the lines BG and AC intersect at E , and let the lines CG and AB intersect at F . Prove that $\angle BPF = \angle CPE$.

Hint: To prove that $\angle BPF = \angle CPE$ given $\angle BPA = \angle CPA$, focus on showing that PA is also the internal angle bisector of $\angle FPE$. Use the properties of the complete quadrilateral $AFBGCE$ to identify harmonic ranges on the lines BC and FE . Project these ranges from P to form harmonic pencils, and recall that a harmonic pencil with a pair of perpendicular conjugate rays must consist of angle bisectors.



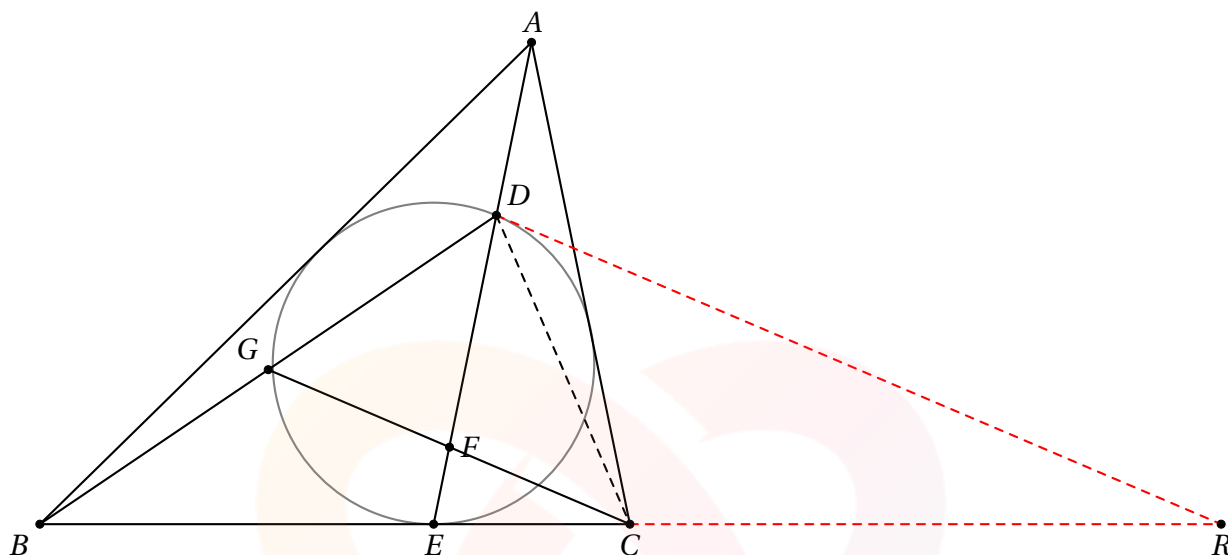
Solution: Let $Q = AP \cap BC$, $H = AP \cap FE$, and $O = BC \cap FE$ (O may lie at infinity). In the complete quadrilateral $AFBGCE$, the concurrency of cevians AQ , BE , and CF at G ensures that $(B, C; Q, O) = -1$ and $(F, E; H, O) = -1$. Projecting the first harmonic range from P yields the harmonic pencil $P(B, C; Q, O) = -1$. Given that $\angle BPA = \angle CPA$ and Q lies on AP , the ray PQ is the internal bisector of $\angle BPC$. A fundamental property of harmonic pencils states that if one ray of a conjugate pair bisects the angle between the other pair, the conjugate rays must be perpendicular; thus, $PQ \perp PO$. Now, projecting the second harmonic range from P gives the harmonic pencil $P(F, E; H, O) = -1$. Since H lies on AP , the ray PH is identical to PQ , and we have established $PH \perp PO$. In this second harmonic pencil, the perpendicularity of the conjugate rays PH and PO implies that PH (and consequently PA) is the internal angle bisector of $\angle FPE$. It follows that $\angle FPA = \angle EPA$, and by subtracting these equal parts from the given equality $\angle BPA = \angle CPA$, we conclude that $\angle BPF = \angle CPE$. ■

Remark

This problem demonstrates the link between harmonicity and orthogonality. A harmonic pencil PB, PC, PQ, PO with PQ bisecting $\angle BPC$ implies PQ is perpendicular to PO . This perpendicularity transfers the bisector property to the second pencil involving FE . PA bisects $\angle FPE$ then finishes the proof.

Problem 53

Let $\triangle ABC$ be a triangle with $AB > AC$. Let its incircle be tangent to side BC at point E . The cevian AE intersects the incircle again at D ($D \neq E$). A point F ($F \neq E$) is chosen on the segment AE such that $CE = CF$. The line CF is extended to intersect BD at point G . Prove that $CF = FG$. (2008 China TST)



Hint: The condition $CE = CF$ creates an isosceles triangle, strongly hinting at an angle chase. Furthermore, because AE connects a vertex to the incircle's point of tangency on the opposite side, it may be related to Gergonne point properties. In Olympiad geometry, a powerful way to leverage the intersection of a cevian and the incircle (D) is to draw the tangent at D to construct a harmonic range on BC .

Solution:

Let the tangent to the incircle at D intersect BC at R . By **Theorem 4.11**, we immediately obtain the harmonic range:

$$(B, C; E, R) = -1$$

Projecting this range from D yields the harmonic pencil $D(B, C; E, R)$.

Since RE and RD are tangents from R to the incircle, $RE = RD$, making $\triangle RDE$ isosceles.

Thus, $\angle RDE = \angle RED$. Given $CE = CF$, $\triangle CEF$ is also isosceles, so $\angle CFE = \angle CEF$.

Since $\angle CEF$ and $\angle RED$ are vertically opposite:

$$\angle CFE = \angle CEF = \angle RED = \angle RDE$$

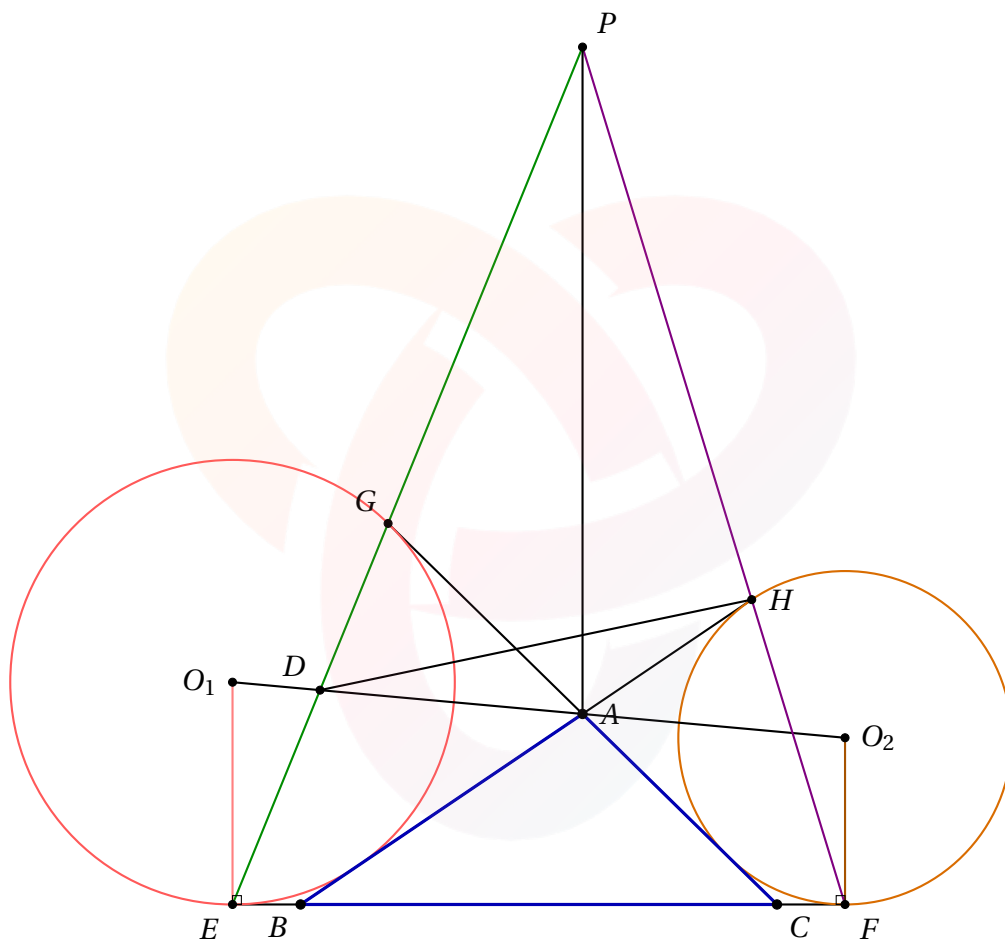
Because the alternate interior angles are equal ($\angle CFE = \angle RDE$), it strictly follows that $CF \parallel DR$.

The line CG intersects the rays DB , DC , and DE of the harmonic pencil $D(B, C; E, R)$ at points G , C , and F , respectively. Because CG is parallel to the fourth ray DR , the segment it intercepts must be bisected by the pencil, thus $CF = FG$. ■

Problem 54

In the diagram below, $\odot O_1$ and $\odot O_2$ are tangent to all three sides of $\triangle ABC$, each with two points of tangency on the extensions. Let E, F, G, H be the points of tangency, and let $EG \cap FH = P$. Prove that $PA \perp BC$.

Hint: Apply Menelaus' Theorem to $\triangle ABD$ and $\triangle ADC$ with suitable transversals. As we want to prove perpendicularity, we can either find cyclic quadrilaterals or prove the existence of a harmonic range.



Solution 1 (Cyclic Quadrilaterals): Connect $O_1 O_2$. Since $\odot O_1$ and $\odot O_2$ are two excircles of $\triangle ABC$, obviously $O_1 O_2$ passes through A . Let $O_1 O_2 \cap EP = D$. So $CE = CG$, $\angle CEG = 90^\circ - \frac{1}{2}\angle C$, $BH = BF$, $\angle BHF = 90^\circ - \frac{1}{2}\angle B$,

and also, $\angle O_1 DE = 180^\circ - \angle ADE = 180^\circ - (360^\circ - \angle DAB - \angle ABE - \angle BED)$

$$= -180^\circ + (90^\circ - \frac{1}{2}\angle A) + (180^\circ - \angle B) + (90^\circ - \frac{1}{2}\angle C)$$

$$= 90^\circ - \frac{1}{2}\angle B = \angle O_1 BE$$

$\Rightarrow O_1 EBD$ is cyclic.

$$\Rightarrow \angle O_1 DB = 180^\circ - \angle O_1 EB = 90^\circ$$

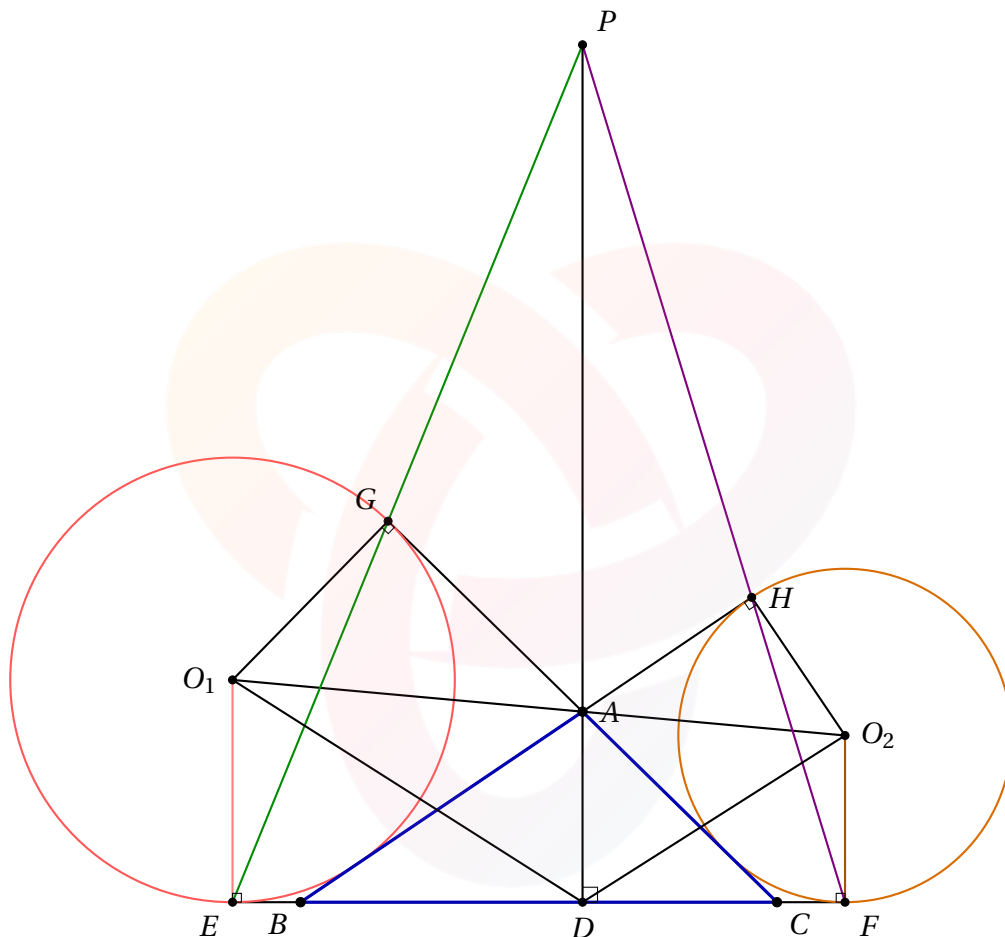
$$\text{Note that } \angle PDA = \angle O_1DE = 90^\circ - \frac{1}{2}\angle B = \angle BHF$$

$\Rightarrow AHPD$ is cyclic.

$$\Rightarrow \angle APH = \angle ADH, \angle O_2HB = \angle O_2FB = \angle O_2DB = 90^\circ.$$

So it follows that B, D, H, O_2, F are concyclic, hence $\angle ADH = \angle O_2FH$.

Now $\angle APH = \angle O_2FH \Rightarrow PA \parallel O_2F$. Since $O_2F \perp BC$, we get $PA \perp BC$. ■



Solution 2: Let $PA \cap BC = D$. We can use Menelaus' Theorem in $\triangle ABD$ with transversal PHF and $\triangle ADC$ with transversal PGE respectively:

$$\begin{aligned} \frac{AH}{HB} \cdot \frac{BF}{FD} \cdot \frac{DP}{PA} &= 1 \\ &= \frac{DP}{PA} \cdot \frac{AG}{GC} \cdot \frac{CE}{ED} \end{aligned}$$

Clearly, from tangency theorems, $BF = HB$, $CE = GC$. Hence

$$\begin{aligned} \frac{AH}{FD} &= \frac{AG}{ED} \\ \Rightarrow \frac{ED}{BF} &= \frac{AG}{AH} \end{aligned}$$

Connecting O_1G and O_2H , from $Rt\triangle AGO_1 \sim Rt\triangle AHO_2$, we get

$$\frac{AG}{AH} = \frac{O_1G}{O_2H}$$

Connecting O_1E and O_2F , we get

$$\frac{AG}{AH} = \frac{O_1E}{O_2F}$$

Connecting O_1D and O_2D , in $Rt\triangle O_1ED$ and $Rt\triangle O_2FD$, there is

$$\frac{ED}{DF} = \frac{O_1E}{O_2F}$$

Hence $Rt\triangle O_1ED \sim Rt\triangle O_2FD$, so $\angle O_1DE = \angle O_2DC$, from which we deduce DF is the external bisector of $\angle O_1DO_2$ in $\triangle O_1DO_2$.

Let $O_1O_2 \cap EF = Q$ (Q may lie at infinity).

Hence $D(O_1, O_2; A, Q) = -1$.

Thus $DA \perp DQ$, so it follows that $PA \perp BC$. ■

Remark

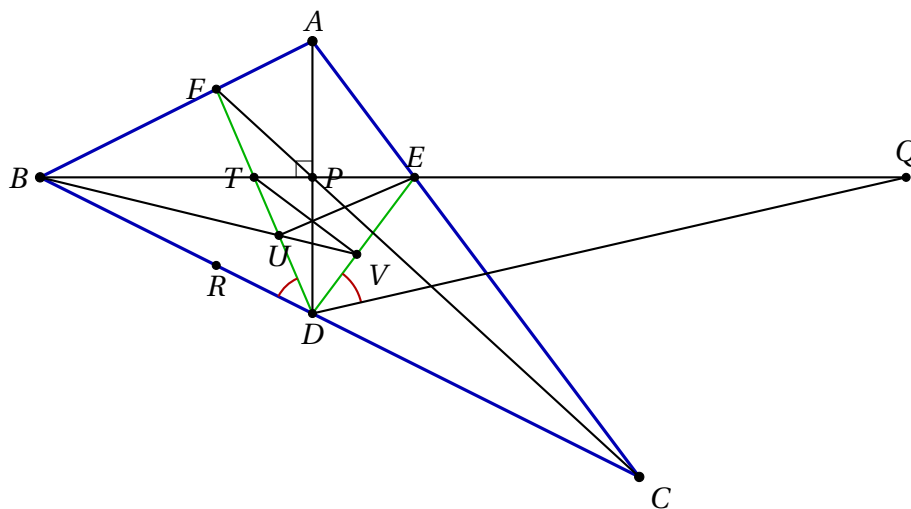
Menelaus' Theorem transitions the problem from triangle sides to circle properties. The proof uses the relationship between harmonic bundles and perpendicularity. Identifying DF as an angle bisector is the main step. Remember to search for harmonic ranges to prove lines are perpendicular and vice versa.

Problem 55

Let P be an interior point of $\triangle ABC$, and define

$$D = AP \cap BC, \quad E = BP \cap CA, \quad F = CP \cap AB.$$

Point Q lies on the extension of BE beyond E and satisfies $\angle EDQ = \angle FDB$. Suppose $BE \perp AD$ and $DQ = 2BD$. Prove that $\angle FDE = 60^\circ$.



Solution: Let $FD \cap BP = T$, and $AT \cap BC = R$.

From Menelaus and Ceva, we obtain:

$$\frac{BC}{DC} = \frac{BF}{AF} \cdot \frac{AP}{RD} = \frac{BR}{DR}.$$

This means B, R, D, C is a harmonic set of points.

Hence AB, AR, AD, AC form a harmonic bundle, so B, T, P, E is a harmonic set of points.

In $\triangle DTE$, let the projections of E, T on DT, DE be U, V respectively.

$$\therefore DP \perp ET$$

$$\therefore EU, TV, DP \text{ are concurrent.}$$

Since B, T, P, E are harmonic, from Ceva we obtain

$$\frac{EV}{VD} \cdot \frac{DU}{UT} = \frac{EP}{TP} = \frac{EB}{BT}.$$

Hence from the converse of Menelaus, B, U, V are collinear.

We know T, U, V, E are concyclic

$$\Rightarrow \angle TUB = \angle TED$$

$$\Rightarrow \angle BUD = \angle QED$$

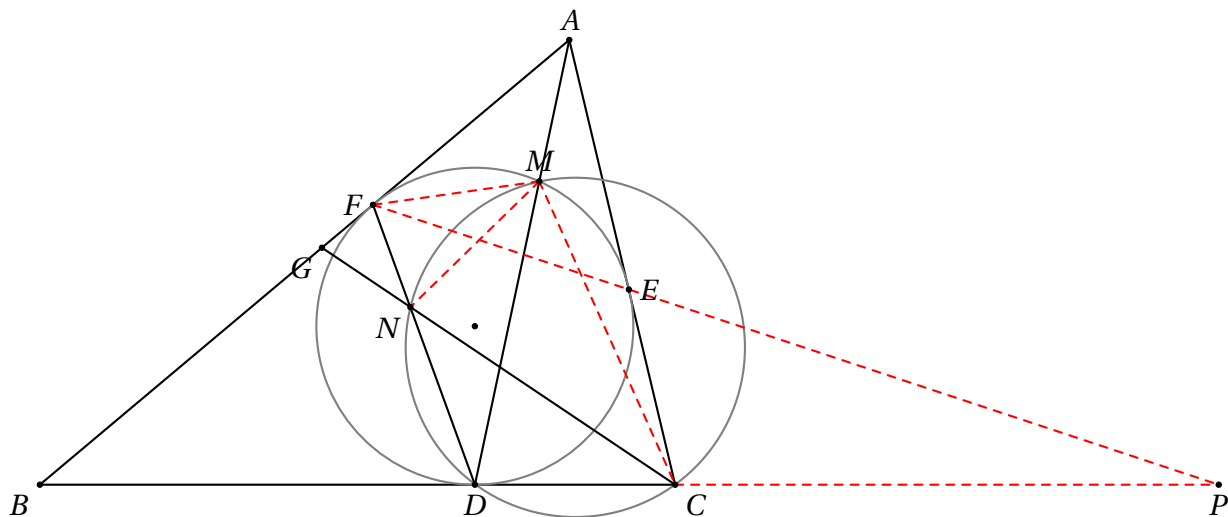
$$\therefore \angle EDQ = \angle FDB$$

$$\therefore \triangle BUD \sim \triangle QED \text{ with similar ratio } \frac{BD}{QD} = \frac{1}{2} \Rightarrow \frac{UD}{ED} = \frac{1}{2}.$$

Also, $EU \perp UD$, hence $\angle FDE = \angle UDE = 60^\circ$. ■

Problem 56

Let the incircle $\odot I$ of $\triangle ABC$ be tangent to sides BC, CA , and AB at points D, E , and F , respectively. The cevian AD intersects $\odot I$ at point M . Let N be the intersection of the circumcircle of $\triangle CDM$ and the line DF . The line CN intersects AB at point G . Prove that $CD = 3FG$.



Solution:

Applying Menelaus's Theorem to $\triangle GBC$ with transversal $F - N - D$ yields:

$$\frac{GF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CN}{NG} = 1$$

Since FB and BD are tangent segments from B to $\odot I$, we have $FB = BD$. This simplifies the relation to $\frac{CN}{NG} = \frac{DC}{GF}$. Therefore, to prove $CD = 3FG$, it suffices to prove $\frac{CN}{NG} = 3$.

To find this ratio, we apply Menelaus's Theorem to $\triangle CND$ with transversal $B - G - F$:

$$\frac{CG}{GN} \cdot \frac{NF}{FD} \cdot \frac{DB}{BC} = 1 \implies \frac{CG}{GN} = \frac{BC}{BD} \cdot \frac{FD}{FN}$$

Because M, N, C, D are concyclic and CD is tangent to $\odot I$ at D , the Alternate Segment Theorem gives $\angle MFD = \angle MDC$. Furthermore, the exterior angle of the cyclic quadrilateral yields $\angle FNM = \angle MCD$. Thus, $\triangle MNF \sim \triangle MCD$. This similarity gives $\frac{FN}{DC} = \frac{MF}{MD}$, or $FN = DC \cdot \frac{MF}{MD}$. Substituting this into our Menelaus ratio yields:

$$\frac{CG}{GN} = \frac{BC}{BD} \cdot \frac{FD}{DC} \cdot \frac{MD}{MF}$$

Notice that AD is the *Gergonne cevian* of $\triangle ABC$. By **Theorem 4.11**, its intersection M with $\odot I$ ensures $MFDE$ is a harmonic quadrilateral. Consequently, by the properties of harmonic quadrilaterals outlined in **Theorem 4.8**, we have $MF \cdot DE = \frac{1}{2} MD \cdot EF$. Rearranging this gives $\frac{MD}{MF} = \frac{2DE}{EF}$. Substituting this into our ratio yields:

$$\frac{CG}{GN} = 2 \left(\frac{BC}{BD} \cdot \frac{FD}{DC} \cdot \frac{DE}{EF} \right)$$

We now evaluate the expression inside the parenthesis by converting the segments into the fundamental side lengths and angles of $\triangle ABC$. We know $BC = a$, $BD = s - b$, and $DC = s - c$. The lengths of the chords of the incircle are $FD = 2(s - b) \sin \frac{B}{2}$, $DE = 2(s - c) \sin \frac{C}{2}$, and $EF = 2(s - a) \sin \frac{A}{2}$. Substituting these in gives:

$$\frac{BC}{BD} \cdot \frac{FD}{DC} \cdot \frac{DE}{EF} = \frac{a}{s - b} \cdot \frac{2(s - b) \sin \frac{B}{2}}{s - c} \cdot \frac{2(s - c) \sin \frac{C}{2}}{2(s - a) \sin \frac{A}{2}} = \frac{a \cdot 2 \sin \frac{B}{2} \sin \frac{C}{2}}{(s - a) \sin \frac{A}{2}}$$

Using the standard circumradius identities $a = 4R \sin \frac{A}{2} \cos \frac{A}{2}$ and $s - a = 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, we substitute these to obtain:

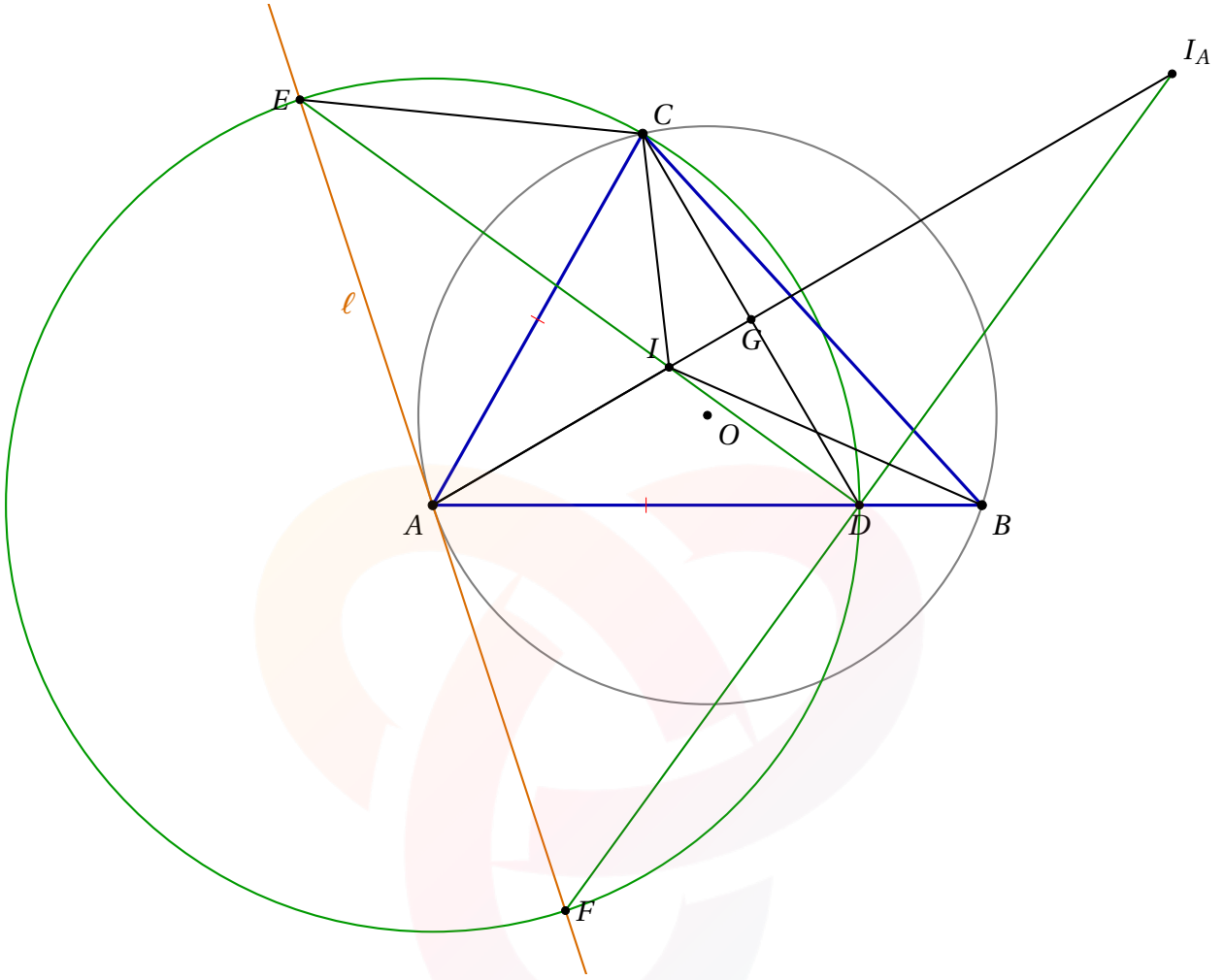
$$\begin{aligned} \frac{(4R \sin \frac{A}{2} \cos \frac{A}{2}) \cdot 2 \sin \frac{B}{2} \sin \frac{C}{2}}{(4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}) \cdot \sin \frac{A}{2}} &= \frac{8R \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 2 \\ \implies \frac{CG}{GN} &= 4 \end{aligned}$$

Since C, N , and G are collinear in that order, $CG = CN + NG \implies CN + NG = 4NG$, hence $CN = 3NG$. As we initially established, this proves $CD = 3FG$. ■

Problem 57

Let $\triangle ABC$ satisfy $AB > AC$, and let ℓ be the tangent to its circumcircle at A . The circle centered at A with radius AC meets AB at D and meets ℓ at E and F . Prove that DE and

DF pass through the incenter I and the A -excenter I_A of $\triangle ABC$, respectively.



Solution 1: Connect DE , DC , and construct the bisector of $\angle BAC$ intersecting DE at I and DC at G , and connect IC .

Since $AD = AC$, it follows that $AG \perp DC$ and $ID = IC$. Also since D, C, E lie on $\odot A$, there is $\angle IAC = \frac{1}{2}\angle DAC = \angle IEC$,

hence A, I, C, E are concyclic $\implies \angle CIE = \angle CAE = \angle ABC$. Also, we know $\angle CIE = 2\angle ICD$, it follows that $\angle ICD = \frac{1}{2}\angle ABC$.

Hence $\angle AIC = \angle IGC + \angle ICG = 90^\circ + \frac{1}{2}\angle ABC$,

$$\angle ACI = 180^\circ - \angle IAC - \angle AIC = \frac{1}{2}\angle ACB$$

Hence I is the incenter of $\triangle ABC$.

Now, let the extension of FD and AI meet at I_A , and connect BI, BI_A .

Hence $\angle DI_AI = 180^\circ - \angle DFA - \angle FAI_A$

$$\begin{aligned} &= 180^\circ - \left(\frac{180^\circ - \angle FAD}{2} \right) - \angle FAD - \frac{1}{2}\angle BAC \\ &= 90^\circ - \frac{1}{2}\angle FAD - \frac{1}{2}\angle BAC \end{aligned}$$

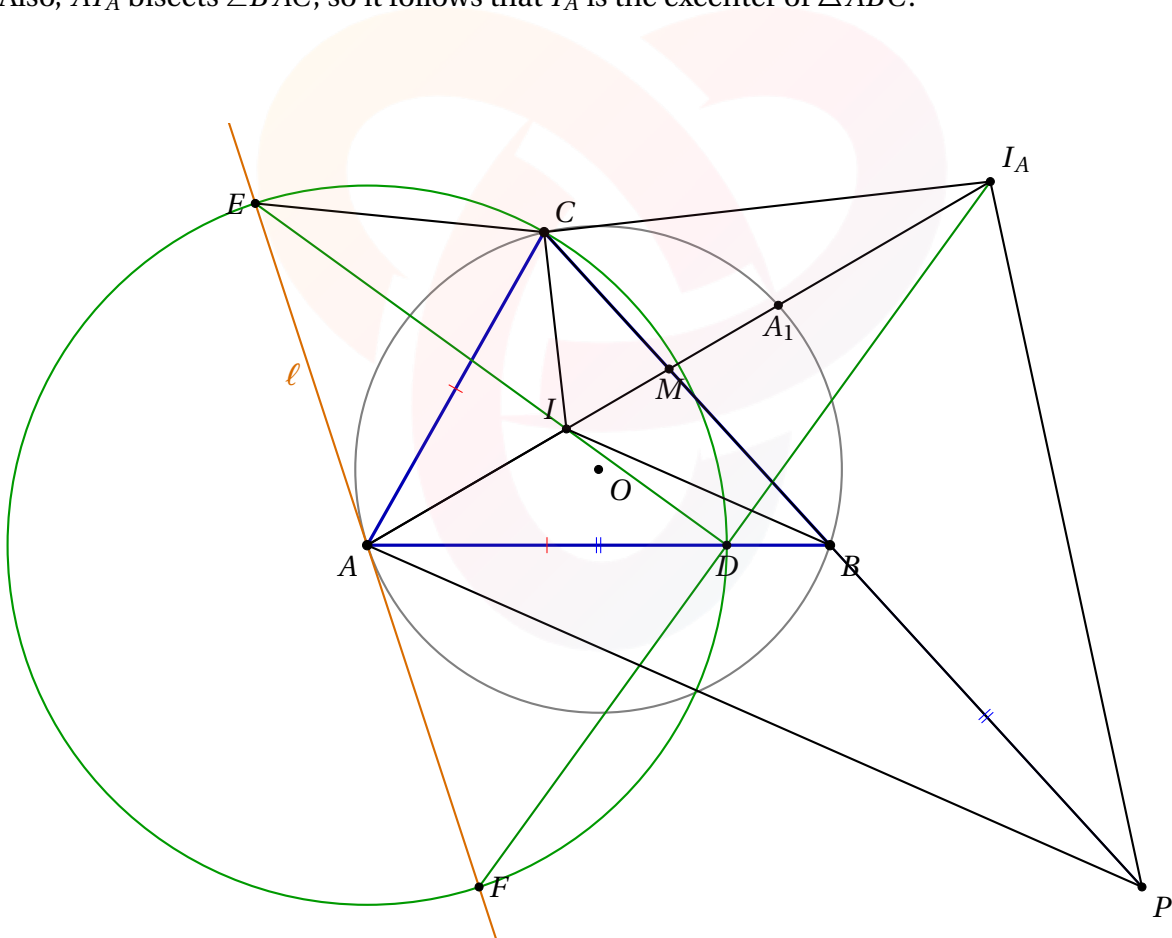
$$\begin{aligned} &= 90^\circ - \frac{1}{2}\angle ACB - \frac{1}{2}\angle BAC \\ &= \frac{1}{2}\angle ABC = \angle DBI \end{aligned}$$

Hence B, D, I, I_A are concyclic. From this we obtain:

$$\begin{aligned}\angle I_A BC + \angle CBI &= \angle I_A BI = \angle I_A DI = 180^\circ - \angle FDA - \angle ADI \\ &= 180^\circ - \frac{1}{2}(180^\circ - \angle FAD) - \angle ACI \\ &= 90^\circ + \frac{1}{2}\angle FAD - \frac{1}{2}\angle ACB = 90^\circ, \\ \therefore \angle I_A BC &= 90^\circ - \angle CBI = 90^\circ - \frac{1}{2}\angle ABC = \frac{1}{2}(180^\circ - \angle ABC)\end{aligned}$$

Hence $I_A B$ bisects the external angle of $\angle ABC$.

Also, AI_A bisects $\angle BAC$, so it follows that I_A is the excenter of $\triangle ABC$.



Solution 2: Construct the bisector of $\angle BAC$, and let it intersect DE at I , so clearly, $\triangle ADI \cong \triangle ACI$.

Hence $\angle ACI = \angle ADE = \frac{1}{2}(180^\circ - \angle EAD) = \frac{1}{2}(180^\circ - \angle BAC - \angle ABC) = \frac{1}{2}\angle ACB$. From this we deduce I is the incenter of $\triangle ABC$.

Let $AI \cap BC = M$, $AI \cap \odot O = A_1$, $AI \cap FD = I_A$, and connect CI_A ,

Hence $\angle DI_A A = \angle AI_A C$.

Extend CB to P such that $PB = BA$,

hence $\angle APC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle B$. Notice $\frac{1}{2}(\angle A + \angle B + \angle C) = 90^\circ = \angle FDA + \angle ADE$

$$= \left(\frac{1}{2}\angle A + \angle DI_A A \right) + \angle ICA = \left(\frac{1}{2}\angle A + \angle AI_A C \right) + \frac{1}{2}\angle C,$$

hence $\angle AI_A C = \frac{1}{2}\angle B = \angle APC$, meaning A, P, I_A, C are concyclic, so

$$\angle AI_A P = \angle ACP = \angle AA_1 B,$$

$$\Rightarrow BA_1 \parallel PI_A, \text{ so it follows that } \frac{I_A A_1}{A_1 M} = \frac{PB}{MB} = \frac{AB}{BM} = \frac{AI}{IM}.$$

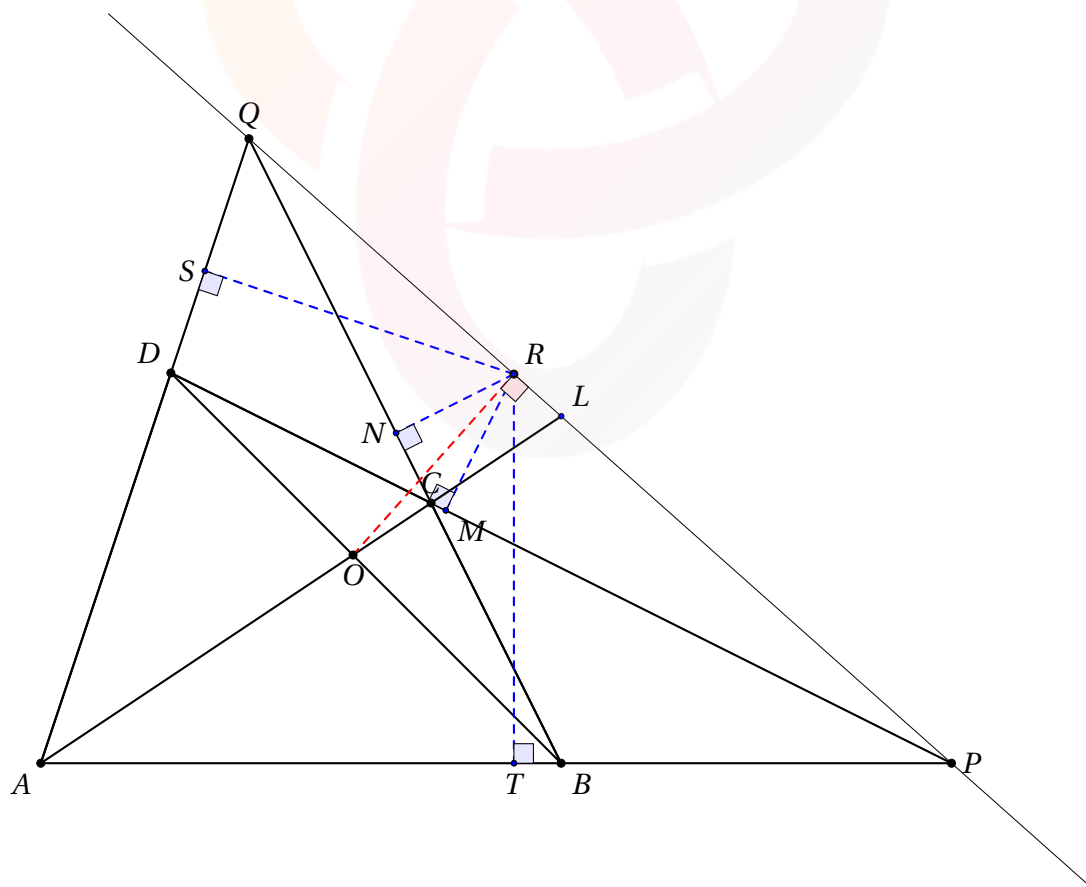
Thus I_A is the A -excenter of $\triangle ABC$. ■

Problem 58

Let $ABCD$ be a convex quadrilateral, and let

$$O = AC \cap BD, \quad P = BA \cap CD, \quad Q = CB \cap DA.$$

Let R be the projection of O onto PQ . Prove that the projections of R onto the lines CD , BC , DA , and AB are concyclic.



Hint: The projections from R creates several cyclic quadrilaterals such as $RTAS$ and $RSNQ$. Our end goal is to prove $MNST$ is cyclic. Consider letting $CA \cap PQ = L$ and finding a relationship between points L, O, A, C . Finally, recall **Theorem 4.8**.

Solution: In the diagram above, let the projection of R onto CD , BC , DA , and AB be M , N , S , and T respectively.

Hence $\angle TSN = \angle TSA + \angle DSN$,

$$\angle TMN = \angle RMN - \angle RMT$$

Since $RTAS$ is cyclic, we obtain $\angle TSA = \angle TRA$.

Similarly, from $RSNQ$ cyclic, we obtain

$$\angle DSN = 180^\circ - \angle NSQ = 180^\circ - \angle NRQ$$

From $RNCM$ and $RPMT$ both cyclic and $\angle CNR = \angle PTR = 90^\circ$, we get

$$\angle RMN = \angle RCN = 90^\circ - \angle CRN, \angle RMT = \angle RPT = 90^\circ - \angle PRT.$$

$$\begin{aligned} \therefore \angle TSN + \angle TMN &= \angle TSA + \angle DSN + \angle RMN - \angle RMT \\ &= \angle TRA + 180^\circ - \angle NRQ + 90^\circ - \angle CRN - (90^\circ - \angle PRT) \\ &= 180^\circ + \angle PRA - \angle QRC \end{aligned}$$

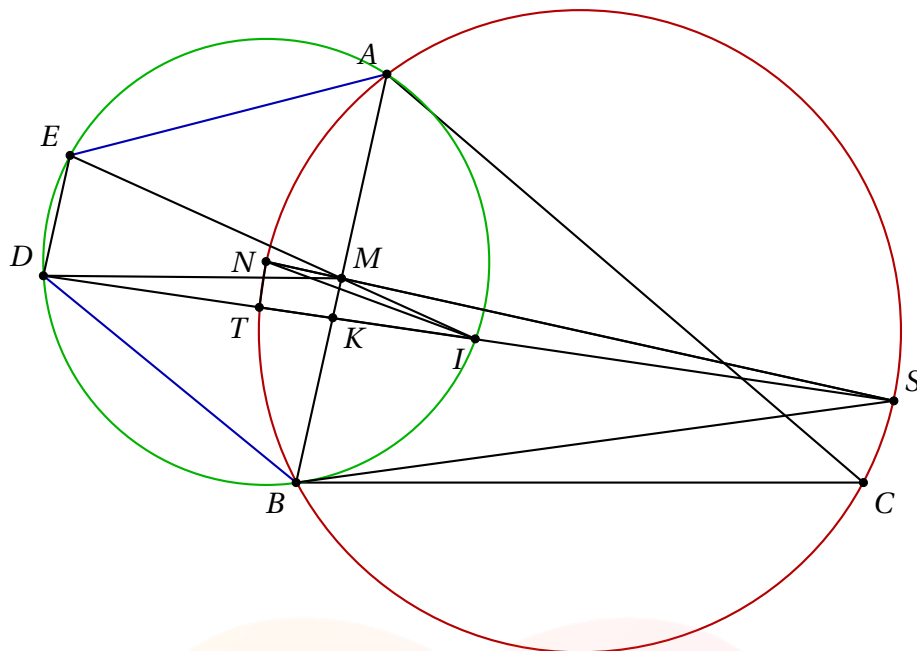
Let $CA \cap PQ = L$. From L, O, A, C are harmonic points and $\angle LRO = 90^\circ$, we deduce that RO and RL are the internal and external bisectors of $\angle ARC$ respectively. Hence $\angle PRA = \angle QRC$. From this, we get $\angle TSN + \angle TMN = 180^\circ$, thus M, N, S, T are concyclic. ■

Remark

- **The Core Lemma:** Note that if points L and O divide a segment AC harmonically (meaning the cross-ratio $(L, O; A, C) = -1$), and a point R forms a right angle $\angle LRO = 90^\circ$, then RL and RO are the internal and external angle bisectors of $\angle ARC$.
- **Complete Quadrilaterals:** The fact that $(L, O; A, C) = -1$ is not a coincidence; it is a fundamental property of the complete quadrilateral formed by the four lines AB, BC, CD , and DA . The line PQ (which contains L) is the polar of the diagonal intersection O .
- **Pedal Circles:** Geometrically, the problem proves that the pedal points of R onto the four lines of the complete quadrilateral are concyclic. Point R is uniquely positioned as the orthogonal projection of O onto its polar line PQ . More on complete quadrilaterals will be mentioned next chapter.

Problem 59

In $\triangle ABC$ with $AC > BC$, let S be the midpoint of the major arc ACB of the circumcircle of $\triangle ABC$, and let I be the incenter. Let SI intersect the circumcircle again at T . Construct point D as the reflection of I in T , and let M be the midpoint of AB . Through D , draw the line parallel to AB , meeting the extension of IM at E . Prove that $AE = BD$.



Solution: Let N be the midpoint of $\widehat{AB}$, hence NMS is collinear and is the perpendicular bisector of $\widehat{AB}$. Let $K = SI \cap AB$.

From $\widehat{SA} = \widehat{SB}$, we get $\angle STB = \angle SBA = \angle SNB$,

hence $\triangle SKB \sim \triangle SBT$, $\triangle SMB \sim \triangle SBN$,

thus $SB^2 = SK \cdot ST = SM \cdot SN$.

Similarly, $NM \cdot NS = NB^2 = NI^2$.

$$\begin{aligned} \therefore TI^2 &= NI^2 - NT^2 \\ &= NM \cdot NS - NS^2 + ST^2 \\ &= ST^2 - SM \cdot SN \\ &= ST^2 - ST \cdot SK \\ &= TK \cdot TS \end{aligned}$$

Now, T is the midpoint of ID , hence it follows that $(D, I; K, S) = -1$.

Also, $\angle KMS = 90^\circ$, so $\angle DMK = \angle IMK$.

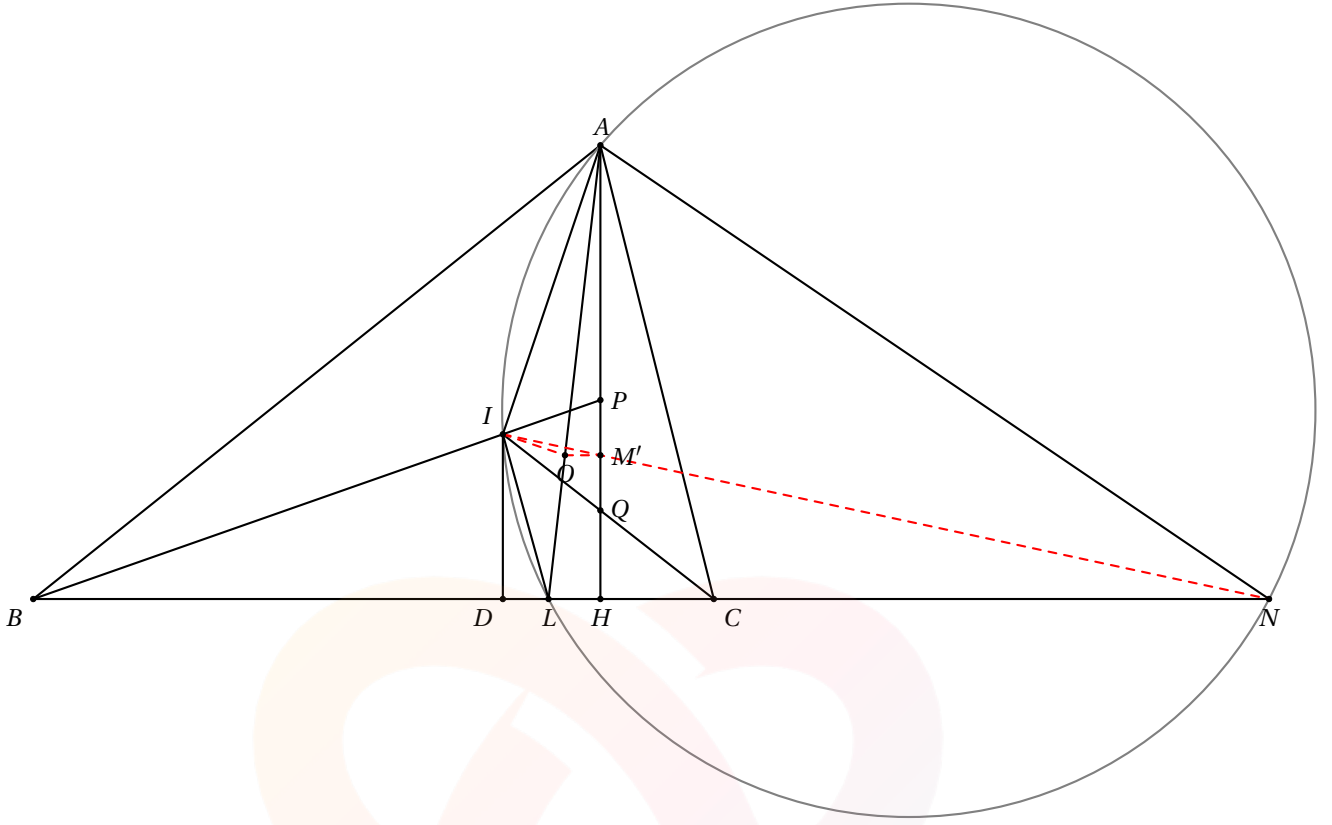
From $DE \parallel AB$, we get $\angle MED = \angle IMK = \angle DMK = \angle EDM$, hence $EM = DM$.

Also, $\angle EMA = \angle DMK$ and $MA = MB$,

so $\triangle MEA \cong \triangle MDB$ (SAS), hence $AE = BD$, and we are done. ■

Problem 60

In $\triangle ABC$ with $AB > AC$, let I be the incenter and D be the projection of I onto BC . The altitude AH from A to BC intersects lines BI and CI at points P and Q , respectively. Let O be the circumcenter of $\triangle IPQ$. The line AO intersects BC at point L . Let N be the second intersection of the circumcircle of $\triangle AIL$ with the line BC . Prove that $\frac{BD}{CD} = \frac{BN}{CN}$. (CGMO 2016)

**Solution:**

To prove $\frac{BD}{CD} = \frac{BN}{CN}$, it suffices to show that the range $(B, C; D, N) = -1$. Projecting through I , this is equivalent to showing the pencil $I(B, C; D, N)$ is harmonic.

We intersect this pencil with the transversal line PQ . Since $ID \perp BC$ and $PQ \perp BC$, the ray $ID \parallel PQ$. By standard properties of harmonic pencils, since the ray ID is parallel to the transversal, the pencil is harmonic if and only if its conjugate ray IN bisects the segment PQ . Letting M' be the midpoint of PQ , it suffices to prove that I, M', N are collinear.

Since O is the circumcenter of $\triangle IPQ$, $OM' \perp PQ \implies OM' \parallel BC \parallel CN$. This parallel relation means I, M', N are collinear if and only if $\angle IM'O = \angle INC$.

Because A, I, L, N are concyclic, $\angle INC = \angle IAL = \angle IAO$. Thus, our collinearity condition reduces to $\angle IM'O = \angle IAO$, which holds if and only if A, I, O, M' are concyclic.

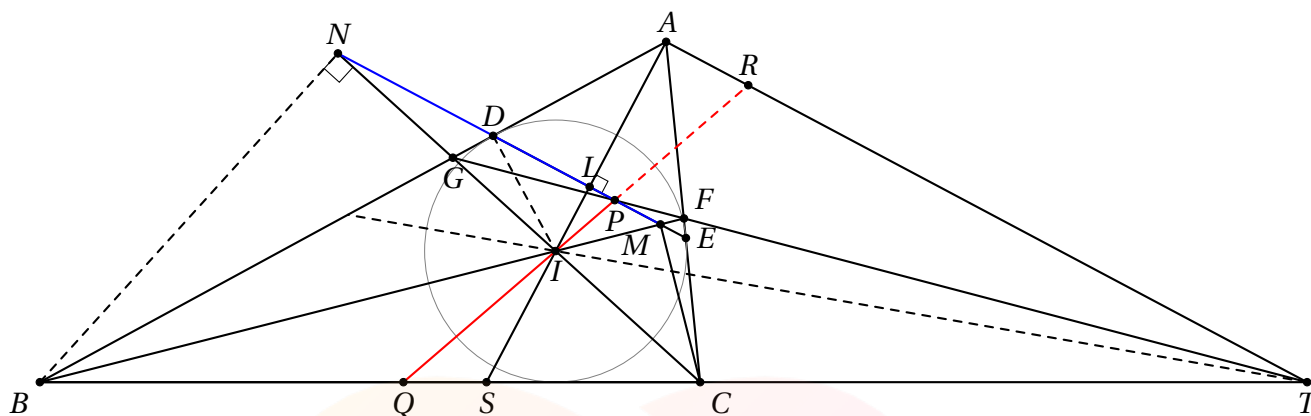
Since $\angle AM'O = 90^\circ$, the points A, I, O, M' are concyclic if and only if the opposite angles sum to 180° , meaning $\angle AIO = 90^\circ$. We evaluate this by splitting the angle $\angle AIO = \angle AIP + \angle OIP$, and computing each of the two angles. First, it is a standard incenter property that $\angle AIB = 90^\circ + \frac{\angle C}{2}$, whose supplement is $\angle AIP = 90^\circ - \frac{\angle C}{2} = 90^\circ - \angle BCI$. Second, in $\odot(IPQ)$ with center O , note that $\angle OIP = 90^\circ - \angle IQP$. Since $\triangle QHC$ is right-angled, we have $\angle IQP = \angle HQC = 90^\circ - \angle BCI$.

$$\therefore \angle AIO = (90^\circ - \angle BCI) + [90^\circ - (90^\circ - \angle BCI)] = 90^\circ$$

We are essentially done here, as this implies A, I, O, M' are concyclic, hence I, M', N are collinear, hence the existence of the harmonic bundle, and hence $\frac{BD}{CD} = \frac{BN}{CN}$. ■

Problem 61

Let D, E be the points of tangency of the incircle of $\triangle ABC$ at sides AB, AC , and let I denote the incenter. Let $F = BI \cap AC$, $G = CI \cap AB$, $M = DE \cap BI$, $N = DE \cap CI$, $P = DE \cap FG$, $Q = BC \cap IP$. Prove that $BC = 2MN$ iff $IQ = 2IP$. (CGMO 2018 P8)



Solution: Consider properties of M, N .

$$M, E, C, I \text{ are concyclic} \implies \angle IMC = \angle IEC = 90^\circ.$$

Similarly, $\angle INB = 90^\circ$.

$$\Rightarrow \angle BMC = \angle BNC = 90^\circ$$

$\Rightarrow B, C, M, N$ are concyclic with diameter BC .

$$\Leftrightarrow \sin \angle MBN = \frac{1}{2}$$

$$\Leftrightarrow \angle MCN = 30^\circ$$

$$\Leftrightarrow \angle MEI = 30^\circ$$

$$\Leftrightarrow \angle DEI = 30^\circ$$

$$\Leftrightarrow \frac{1}{2} \angle A = 30^\circ$$

$$\Leftrightarrow \angle A = 60^\circ$$

Now it suffices to prove $IQ = 2IP$ **iff** $\angle A = 60^\circ$.

Method 1: Consider trigonometric methods of simplifying $\frac{IP}{IQ}$. This is left as a challenge for the reader, as I will provide a cross ratio implementation. Method 2: Relate $P = DE \cap FG \rightarrow Q$. Note that P lies on GF . Let $GF \cap BC = T$, $AI \cap BC = S$.

$$\implies (B, C; S, T) = -1$$

Also, AS bisects $\angle BAC$, so it follows that $\angle SAT = 90^\circ$ (**Theorem 4.2 part 4**).

Now, we let $L = AI \cap DE$, and let $R = AT \cap IP$.

From complete quadrilateral $TFGIBC$, we obtain the following: $T(B, G; I, A) = -1$ and $(Q, P; I, R) = -1$

$$\Rightarrow \frac{IP}{IO} = \frac{RP}{RO}$$

Notice that $\frac{IP}{IR} = \frac{IL}{IA} = \frac{ID^2}{AD^2} = \tan^2 \frac{A}{2}$ (only focuses on I, P, R , ignores Q)

$$\text{Hence } \frac{IP}{IQ} = \frac{1}{2} \Leftrightarrow \frac{IP}{PR} = \frac{1}{3}$$

$$\Leftrightarrow \tan^2 \frac{A}{2} = \frac{1}{3}$$

$$\Leftrightarrow \angle A = 60^\circ$$

