

KMO Solutions - Kevin Hu

Problem 1

Find all pairs of primes (p, q) such that $p^q + q^p$ is also prime.

Solution:

By symmetry, WLOG $p \leq q$. If both p and q are odd primes, then p^q and q^p are both odd. Their sum $p^q + q^p$ would then be an even number greater than 2, which cannot be prime. Thus, at least one of the primes must be 2. Let $p = 2$. We seek odd primes q such that $2^q + q^2$ is prime.

If $q = 3$, we have $2^3 + 3^2 = 17$, which is prime. This gives the solutions $(2, 3)$ and $(3, 2)$. Now consider $q > 3$. Since q is prime and $q > 3$, we have $q \equiv 1 \pmod{3}$ or $q \equiv 2 \pmod{3}$. In either case, $q^2 \equiv 1 \pmod{3}$. For the term 2^q , since q is an odd prime, $q = 2k + 1$, so $2^q = 2 \cdot 4^k \equiv 2 \pmod{3}$. Summing these, we obtain:

$$2^q + q^2 \equiv 2 + 1 \equiv 0 \pmod{3}$$

Since $q > 3$, $2^q + q^2 > 3$, so the sum is a multiple of 3 and composite.

Finally, we check the case where $q = 2$. If $p = 2$ and $q = 2$, then $p^q + q^p = 2^2 + 2^2 = 8$, which is not prime.

The only pairs (p, q) are $(2, 3)$ and $(3, 2)$.

Problem 2

Let M be the set of points (x, y) in the Cartesian plane such that $x \in \{1, 2, 3, 4, 5\}$ and $y \in \{1, 2, 3, 4\}$. Suppose that 11 of these points are colored red. Prove that there must exist four red points which form the vertices of a rectangle with sides parallel to the coordinate axes.

Solution: We will repeatedly use Extended Pigeonhole Principle. If 11 points out of 20



Figure 1: 4x5 grid of nodes for set M

are painted red, there must be at least one row with at least 2 points. WLOG let that be row 4. This means there should be 9 red points in a 15 point rectangle. So there has to be at least one column with at least 1 red point. Let that column be no.5. This means there should be 8 red points in a 12 point rectangle. So there has to be at least one row with at least 2 red points. Let that row be no.3. So there are 6 red points in a 8 point rectangle (2 by 4), so by EPP, $6-4=2$, so there must exist a rectangle where all 4 vertices are red.

Alternative Solution: Let n_i be the number of red points in column i , for $i \in \{1, 2, 3, 4, 5\}$. We are given $\sum_{i=1}^5 n_i = 11$. A rectangle is formed if two columns share red points in the same two rows. The number of pairs of red points in column i is $\binom{n_i}{2}$. The total number of such pairs across all columns is $S = \sum_{i=1}^5 \binom{n_i}{2}$.

By the convexity of the binomial coefficient, S is minimized when the n_i are as nearly equal as possible. Since $11 = 2 \times 4 + 3$, the minimum value occurs when four columns have 2 red points and one column has 3:

$$S \geq 4 \binom{2}{2} + \binom{3}{2} = 4(1) + 3 = 7$$

There are $\binom{4}{2} = 6$ possible pairs of rows in the grid. Since there are at least 7 pairs of red points distributed among only 6 possible row-pair positions, by the Pigeonhole Principle, at least two pairs must occupy the same two rows. These two pairs in their respective columns form the vertices of the required rectangle.



Problem 3

Let m and n be positive integers. Prove that the sum

$$5^n + 5^m$$

can be expressed as the sum of two perfect squares **if and only if** $n - m$ is even.

Solution:

Since $5^n + 5^m$ is symmetrical, we may WLOG $n \geq m$.

If both n and m are even, both 5^n and 5^m are perfect squares, hence satisfying the conditions.

If n and m are both odd, let $n = 2k + 1$ and $m = 2j + 1$. Then:

$$\begin{aligned} 5^n + 5^m &= 5^m(5^{n-m} + 1) \\ &= 5^{2j} \cdot 5(5^{2(k-j)} + 1) \\ &= 5^{2j}(4 + 1)(5^{2(k-j)} + 1) \\ &= 5^{2j}(4 \cdot 5^{2(k-j)} + 5^{2(k-j)} + 4 + 1) \\ &= 5^{2j}[(2 \cdot 5^{k-j})^2 + 2(2 \cdot 5^{k-j}) \cdot 1 + 1^2 + (5^{k-j})^2 - 2(5^{k-j} \cdot 2) + 2^2] \\ &= 5^{2j}[(2 \cdot 5^{k-j} + 1)^2 + (5^{k-j} - 2)^2] \\ &= (5^j(2 \cdot 5^{k-j} + 1))^2 + (5^j(5^{k-j} - 2))^2 \end{aligned}$$

which is the sum of two perfect squares.

However, if one of n, m is odd and the other is even, WLOG let n be even and m be odd (e.g., $n = 2k, m = 2k + 1$). Since $5^2 \equiv 1 \pmod{8}$, then:

$$5^n \equiv 1 \pmod{8} \quad \text{and} \quad 5^m \equiv 5 \pmod{8}$$

Summing these gives:

$$5^n + 5^m \equiv 1 + 5 \equiv 6 \pmod{8}$$

Since the sum of two perfect squares must be congruent to 0, 1, 2, 4, or 5 $\pmod{8}$, $5^n + 5^m$ cannot be expressed as the sum of two perfect squares when n and m differ in parity. $\square$

Problem 4

Let $\triangle ABC$ be an acute-angled triangle such that $AB > AC$. Let M be the midpoint of the side BC . A point P is chosen inside the triangle AMC such that

$$\angle BAM = \angle PAC.$$

Let O, O_1 , and O_2 denote the circumcenters of $\triangle ABC, \triangle ABP$, and $\triangle ACP$, respectively.

Prove that the line AO bisects the segment O_1O_2 .

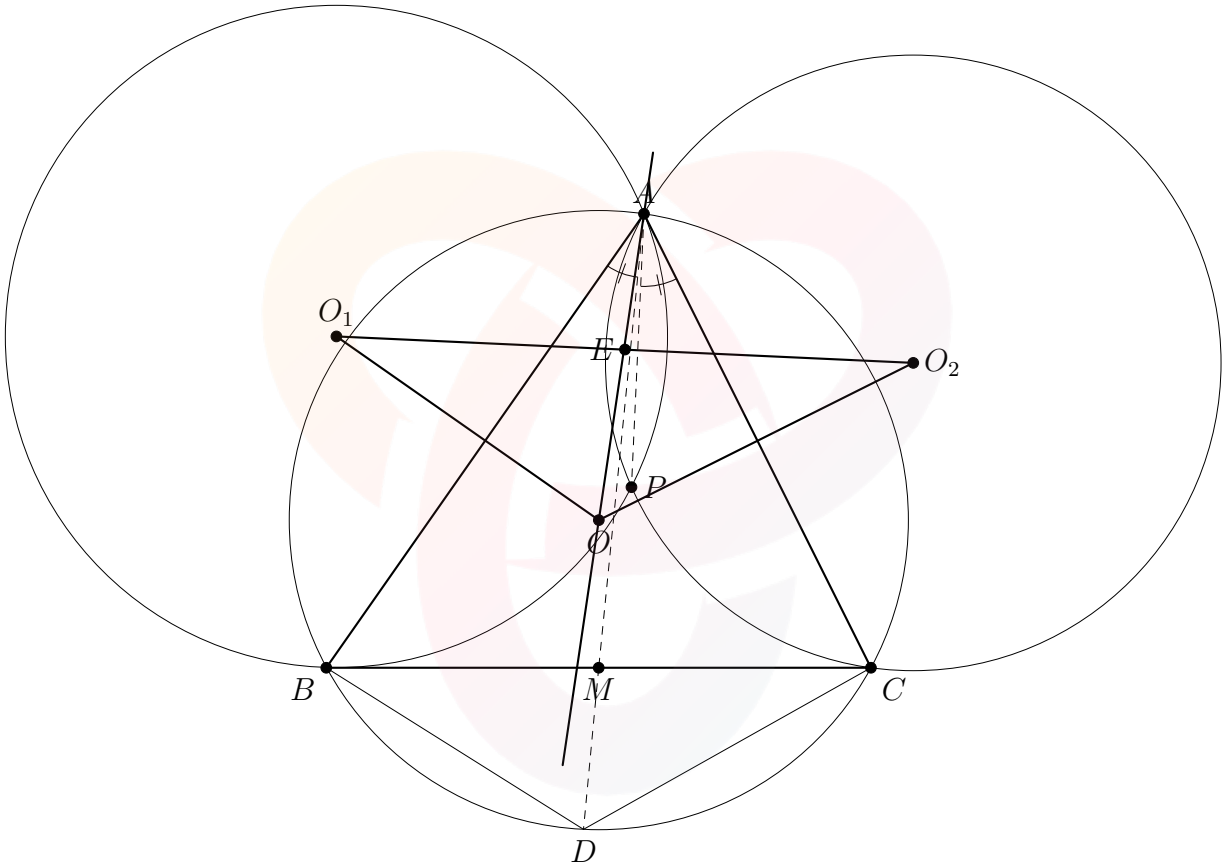


Figure 2: $\triangle ABC, \triangle ABP, \triangle ACP$ and its circumcenters.

Motivation: It is not too difficult to realize that $\angle AO_1O_2 = \angle ABC$. Using the given information and some angle chasing, we can obtain $\angle O_2 = \angle BAM$. Then, using the fact that M is the midpoint of BC , we can identify similar triangles to prove that E is the midpoint of O_1O_2 .

Solution: Since O_1 and O_2 are the circumcenters of $\triangle ABP$ and $\triangle ACP$ respectively, $O_1O_2 \perp AP$. Similarly, $O_1O_3 \perp AC$, which implies $\angle O_2 = \angle PAC$. Given that $\angle BAM = \angle PAC$, we obtain:

$$\angle O_2 = \angle BAM$$

Since O is the circumcenter of $\triangle ABC$, we have $\angle O_2OE = \angle ABM$. This implies:

$$\triangle O_2OE \sim \triangle ABM$$

Similarly:

$$\triangle O_1OE \sim \triangle ACM$$

From these similarities, we obtain:

$$\frac{O_1E}{OE} = \frac{AM}{CM} \quad \text{and} \quad \frac{O_2E}{OE} = \frac{AM}{BM}$$

Since M is the midpoint of BC , it follows that $BM = CM$. Substituting this into our ratios, we obtain:

$$O_1E = O_2E$$

Thus, the line AO bisects O_1O_2 . □



Alternative: Extend AM until it meets the circumcircle of triangle ABC at D , and connect BD and CD .

From the properties of isogonal conjugates and cyclic quadrilateral $ABDC$, we have $\angle AO_1O_2 = \angle MAC = \angle CBD$. Similarly, we have $\angle AO_2O_1 = \angle BAM = \angle BCD$.

Consequently, we identify a similarity between the triangle of circumcenters and the triangle formed by the extension:

$$\triangle O_1O_2O \sim \triangle BCD$$

From this, it is not too difficult to realise that $\triangle OEO_2$ and $\triangle DMC$ are similar, and $\triangle OEO_1$ and $\triangle DMB$ are similar.

Therefore, we obtain:

$$\frac{O_1E}{BM} = \frac{O_1O_2}{BC}$$

Since M is the midpoint of BC , we have $BM = \frac{1}{2}BC$. So $O_1E = \frac{1}{2}O_1O_2$, it follows that E is the midpoint. Thus, AO bisects O_1O_2 . $\square$

Problem 5

Prove that $\forall a, b, c \geq 0$:

$$\frac{a}{\sqrt{4b^2 + 4c^2 + bc}} + \frac{b}{\sqrt{4c^2 + 4a^2 + ca}} + \frac{c}{\sqrt{4a^2 + 4b^2 + ab}} \geq 1$$

Solution:

By Holder's Inequality in the form $(\sum \frac{a}{f})^2 (\sum af^2) \geq (\sum a)^3$, we have:

$$\left(\sum_{cyc} \frac{a}{\sqrt{4b^2 + 4c^2 + bc}} \right)^2 \geq \frac{(a+b+c)^3}{\sum_{cyc} a(4b^2 + 4c^2 + bc)}$$

Hence, it suffices to show:

$$(a+b+c)^3 \geq \sum_{cyc} a(4b^2 + 4c^2 + bc)$$

Expanding gives:

$$\begin{aligned} \sum_{cyc} a^3 + 3 \sum_{sym} a^2b + 6abc &\geq 4 \sum_{sym} a^2b + 3abc \\ \Rightarrow \sum_{cyc} a^3 + 3abc &\geq \sum_{sym} a^2b \end{aligned}$$

Notice this is the standard form of Schur's Inequality for $n = 1$, which states:

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

holds $\forall a, b, c \geq 0$, with equality at $a = b = c$ or when two variables are equal and the third is zero. $\square$

Problem 6

Let Γ denote the circumcircle of $\triangle ABC$. Let L_1 be the tangent to Γ at point C . Let L_2 be the line passing through B parallel to AC . Let D be the intersection of L_1 and L_2 .

Let L_3 be the tangent to Γ at point B . Let L_4 be the line passing through C parallel to AB . Let E be the intersection of L_3 and L_4 .

Let L be the intersection of BE and CD .

Let Γ_1 be the circumcircle of $\triangle BDC$. Let T be the intersection of Γ_1 and line AC such that $T \neq C$.

Let Γ_2 be the circumcircle of $\triangle BEC$. Let S be the intersection of Γ_2 and line AB such that $S \neq B$.

Prove that the ST , BC , and AL are concurrent.

Solution: From $BD \parallel AC$, we get $\angle DBC = \angle ACB$ (alternate angles equal). Also, $\angle BCD = \angle BAC$, so

$$\begin{aligned}\triangle ABC &\sim \triangle CDB(AA) \\ \Rightarrow \angle BDC &= \angle ABC\end{aligned}$$

$\Rightarrow AB$ is tangent to circle Γ_1 at B .

Similarly, AC is tangent to circle Γ_2 at C .

It follows that $\angle ABT = \angle ACB$, $\angle BSC = \angle ACB \Rightarrow BT \parallel SC$.

Let X be the intersection of ST and BC . From trapezoid $BTCS$, we obtain: $\frac{BX}{XC} = \frac{BT}{SC}$, and from $\triangle ABT \sim \triangle ASC \Rightarrow \frac{BT}{SC} = \frac{AB}{AS}$, hence

$$\frac{BX}{XC} = \frac{AB}{AS}$$

From power of a point theorem, $AC^2 = AB \cdot AS \Rightarrow AS = \frac{AC^2}{AB}$. Substituting, we get

$$\frac{BX}{XC} = \frac{AB^2}{AC^2}$$

Thus, X lies on the A-symmedian of $\triangle ABC$, and since LB and LC are both tangent to Γ , AL is the A-symmedian of $\triangle ABC$. Hence, X lies on AL , meaning ST , BC , AL are concurrent (at X). $\square$

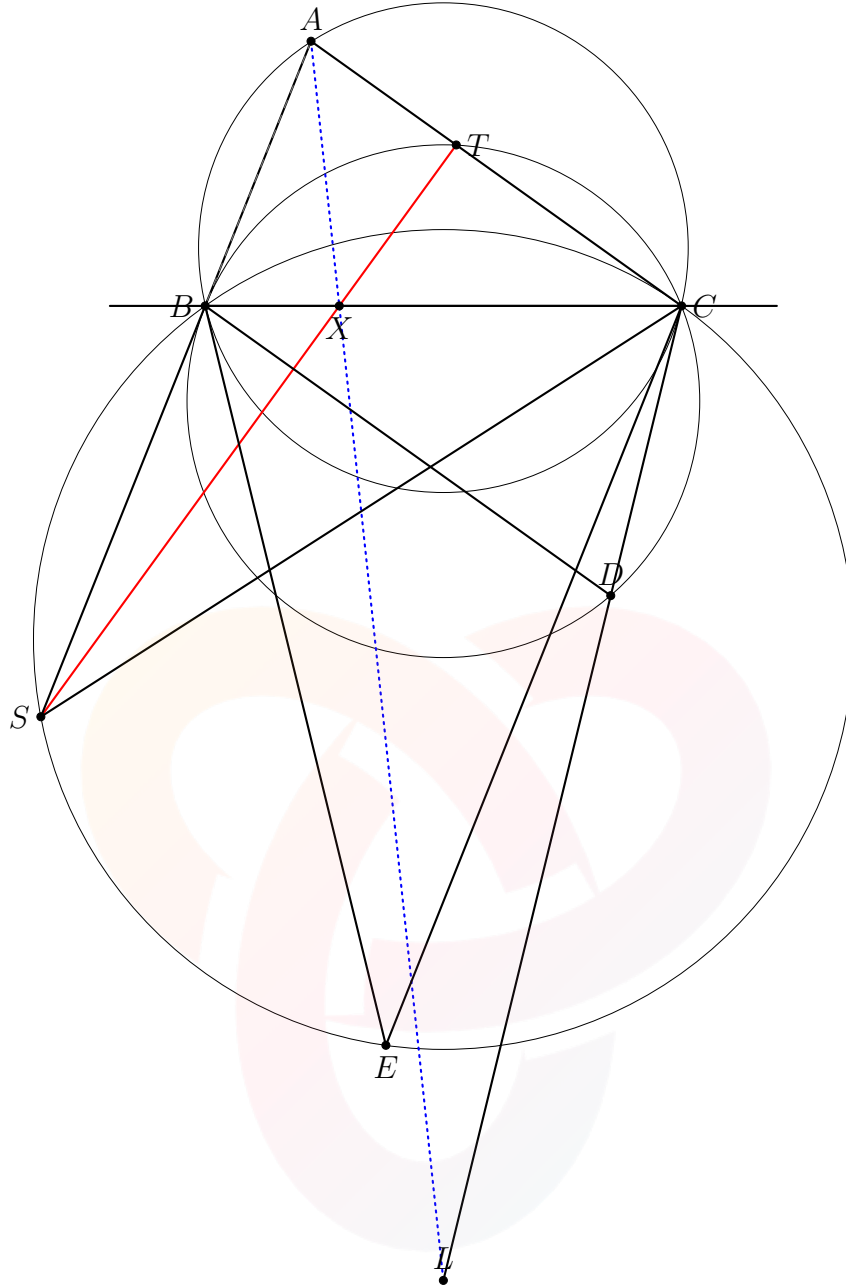


Figure 3: Concurrent lines ST , AL and BC meet at X .

Problem: Prove that there exist infinitely many pairs of distinct positive integers (x, y) such that $x^2 + y^3$ divides $x^3 + y^2$.

Solution:

We construct an infinite family of solutions by setting $x = my$ for some integer $m > 1$. The condition $x^2 + y^3 \mid x^3 + y^2$ becomes:

$$(my)^2 + y^3 \mid (my)^3 + y^2 \implies y^2(m^2 + y) \mid y^2(m^3y + 1)$$

This reduces to the linear condition in y :

$$m^2 + y \mid m^3y + 1$$

Using the property that $m^2 + y \equiv 0 \pmod{m^2 + y}$, we substitute $y \equiv -m^2$:

$$m^3(-m^2) + 1 = 1 - m^5 \equiv 0 \pmod{m^2 + y}$$

To satisfy this, we choose y such that $m^2 + y$ is a divisor of $m^5 - 1$. By setting $m^2 + y = m^5 - 1$, we obtain the general forms:

$$y = m^5 - m^2 - 1 \quad \text{and} \quad x = my = m^6 - m^3 - m$$

For any $m \geq 2$, x and y are distinct positive integers. Hence the quotient becomes:

$$\frac{m^3(m^5 - m^2 - 1) + 1}{m^5 - 1} = \frac{m^8 - m^5 - m^3 + 1}{m^5 - 1} = \frac{(m^5 - 1)(m^3 - 1)}{m^5 - 1} = m^3 - 1$$

Since the quotient $m^3 - 1$ is an integer for all m , and there are infinitely many such m , the proof is complete. $\square$

