

HU Collection TMUA Mock Set C Paper 1

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	C	E	A	D	F	C	E	D	A	E

Q	11	12	13	14	15	16	17	18	19	20
Ans	F	C	A	B	D	B	F	E	C	B

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer C.

Let $t = \log_a b$. Then $\log_b a = \frac{1}{t}$, so $t = \frac{2}{t}$ and $t^2 = 2$. Since $a, b > 1$, $t > 0$ and $t = \sqrt{2}$. Now

$$\log_{ab} a = \frac{1}{\log_a(ab)} = \frac{1}{1 + \log_a b} = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1.$$

Option D, $2 - \sqrt{2}$, is $\log_{ab} b = \frac{t}{1+t}$: the same computation with the roles of a and b swapped.

Question 2. Answer E.

The remainder has degree at most 2. Since dividing $r(x)$ by $(x-1)^2$ must leave the same remainder as dividing $p(x)$ by $(x-1)^2$, we can write $r(x) = c(x-1)^2 + 2x + 1$ for some constant c . The second condition says $r(-1) = p(-1) = -3$, so $4c - 1 = -3$ and $c = -\frac{1}{2}$. Then

$$r(3) = -\frac{1}{2} \cdot 4 + 7 = 5.$$

The whole question is the choice of form for r : writing $r(x) = \alpha x^2 + \beta x + \gamma$ and using the derivative condition at $x = 1$ also works, but takes three equations instead of one. Option F comes from expanding $-\frac{1}{2}(x-1)^2$ with a sign slip.

Question 3. Answer A.

Put $t = |x| \geq 0$. The inequality is $-2 < t^2 - 5t + 4 < 2$. The left half is $t^2 - 5t + 6 > 0$, so $t < 2$ or $t > 3$; the right half is $t^2 - 5t + 2 < 0$, so $\frac{5-\sqrt{17}}{2} < t < \frac{5+\sqrt{17}}{2}$, roughly $0.44 < t < 4.56$. Among non-negative integers this leaves $t = 1$ and $t = 4$. The boundary values $t = 2$ and $t = 3$ give $|t^2 - 5t + 4| = 2$ exactly and are excluded by the strict inequality, and $t = 0$ gives 4. So $x \in \{\pm 1, \pm 4\}$: four integers. With $\leq$ instead of $<$ the answer would be 8.

Question 4. Answer D.

Split the $\frac{16}{x}$ so that the product of the terms is constant: $x^2 + \frac{8}{x} + \frac{8}{x} \geq 3\sqrt[3]{x^2 \cdot \frac{8}{x} \cdot \frac{8}{x}} = 3\sqrt[3]{64} = 12$, with equality when $x^2 = \frac{8}{x}$, that is $x = 2$. Calculus agrees: the derivative $2x - \frac{16}{x^2}$ vanishes at $x = 2$, where the value is $4 + 8 = 12$. Applying AM-GM to the two terms x^2 and $\frac{16}{x}$ directly gives the bound $8\sqrt{x}$, which is not a constant and so says nothing about the minimum.

Question 5. Answer F.

$\tan x + \cot x = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{2}{\sin 2x}$. For $0 < x < \frac{\pi}{2}$, $\sin 2x > 0$, so the inequality is $\sin 2x > \frac{1}{2}$, that is $\frac{\pi}{6} < 2x < \frac{5\pi}{6}$, or $\frac{\pi}{12} < x < \frac{5\pi}{12}$. The length is $\frac{\pi}{3}$. The identity is the whole question; clearing denominators directly leads to a quadratic in $\tan x$ that gets to the same place more slowly.

Question 6. Answer **C**.

The series is geometric with ratio $2 \sin x$, so it converges only when $|2 \sin x| < 1$, and then its sum is $\frac{1}{1-2 \sin x}$. Setting this equal to k gives $2 \sin x = 1 - \frac{1}{k}$, and convergence requires $|1 - \frac{1}{k}| < 1$, that is $0 < \frac{1}{k} < 2$, or $k > \frac{1}{2}$. Any such k gives $|\sin x| < \frac{1}{2}$, so a real x exists. Hence $k = \frac{2}{3}, 2, 4$ work and $k = \frac{1}{3}, \frac{1}{2}$ do not: three values. The trap is $k = \frac{1}{3}$, where solving the formula gives $\sin x = -1$, a perfectly good value of sine at which the series diverges.

Question 7. Answer **E**.

Put $s = \sin x \in [-1, 1]$. A value of s strictly between -1 and 1 gives two values of x in $[0, 2\pi)$; $s = \pm 1$ gives one each. Write the equation as $k = g(s) = -2s^2 - 3s$. On $[-1, 1]$, g rises from $g(-1) = 1$ to its maximum $g(-\frac{3}{4}) = \frac{9}{8}$, then falls to $g(1) = -5$. For an integer k with $-5 < k < 1$, the horizontal line meets the falling branch exactly once, at some s strictly inside $(-1, 1)$, and misses the rising branch: exactly two solutions. At $k = 1$ the roots are $s = -1$ and $s = -\frac{1}{2}$, giving $1 + 2 = 3$ solutions; at $k = -5$ the only admissible root is $s = 1$, giving 1. So $k \in \{-4, -3, -2, -1, 0\}$: five integers.

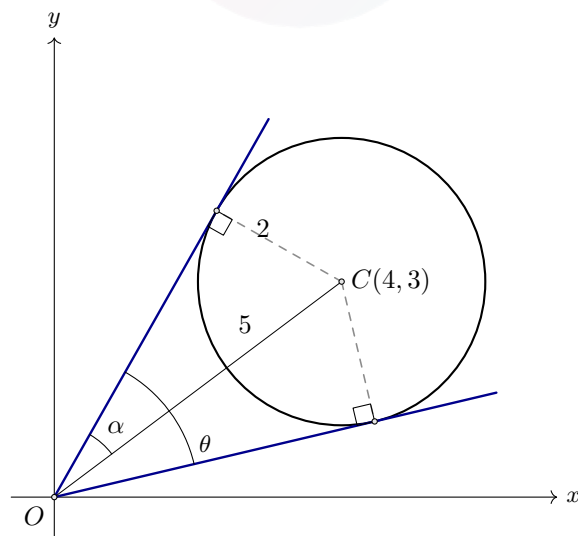
Question 8. Answer **D**.

Pair the outer terms: $\sin x + \sin 3x = 2 \sin 2x \cos x$, so the expression is $\sin 2x (2 \cos x + 1)$. On $[0, 2\pi]$, $\sin 2x$ changes sign at $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and $2 \cos x + 1$ changes sign at $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Checking signs on the six intervals, the product is positive on $(0, \frac{\pi}{2})$, $(\frac{2\pi}{3}, \pi)$ and $(\frac{4\pi}{3}, \frac{3\pi}{2})$, of total length $\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{6} = \pi$. If the sum-to-product step is unfamiliar, expand $\sin 3x = 3 \sin x - 4 \sin^3 x$ and $\sin 2x = 2 \sin x \cos x$ to reach the same factorisation.

Question 9. Answer **A**.

Let $g(x) = x^3 - 3ax^2 + 4$, so $g'(x) = 3x(x - 2a)$. If $a = 0$, g is increasing and has exactly one root. Otherwise the stationary values are $g(0) = 4$ and $g(2a) = 4 - 4a^3$, and there is exactly one real root precisely when these have the same sign, that is $4 - 4a^3 > 0$, or $a < 1$. At $a = 1$, $g(x) = (x - 2)^2(x + 1)$ has two distinct roots, so $a = 1$ is excluded. Hence $a < 1$, including $a = 0$. Option B is the answer to a different question, the one that also demands two stationary points.

Question 10. Answer **E**.



The centre $C = (4, 3)$ is at distance 5 from the origin O and the radius is 2. Each tangent makes an angle α with OC where $\sin \alpha = \frac{2}{5}$, so $\tan \alpha = \frac{2}{\sqrt{21}}$, and the angle between the tangents is $\theta = 2\alpha$,

which is acute since $\alpha < \frac{\pi}{4}$. Then

$$\tan \theta = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{4/\sqrt{21}}{17/21} = \frac{4\sqrt{21}}{17}.$$

The coordinate route, requiring $y = mx$ to be at distance 2 from C , gives $12m^2 - 24m + 5 = 0$ and reaches the same value through $\frac{m_1 - m_2}{1 + m_1 m_2}$, with more work. Option B is $\tan \alpha$, the half-angle.

Question 11. Answer **F**.

The variable x appears inside the integrand, so take it outside first:

$$F(x) = x \int_0^x \sin t \, dt - \int_0^x t \sin t \, dt = x(1 - \cos x) - (\sin x - x \cos x) = x - \sin x.$$

So $F(\pi) = \pi$. The integration by parts is routine; the step that matters is noticing that x is a constant as far as the t -integration is concerned and pulling it out.

Question 12. Answer **C**.

Substitute $u = 3x + 1$: $dx = \frac{1}{3}du$, and the limits become 1 and 2026, so the integral is $\frac{1}{3} \int_1^{2026} f(u) \, du$. The interval $[1, 2026]$ has length $2025 = 3 \times 675$, exactly 675 periods, and the integral of a periodic function over any interval of one period is the same, here 2. So the integral is $\frac{1}{3} \cdot 675 \cdot 2 = 450$. Option E forgets the $\frac{1}{3}$ from dx .

Question 13. Answer **A**.

Let $g(x) = \frac{1}{1+2^{x-2}}$ and pair x with $4 - x$. Writing $t = 2^{x-2}$, we have $2^{(4-x)-2} = \frac{1}{t}$, so

$$g(x) + g(4 - x) = \frac{1}{1+t} + \frac{1}{1+\frac{1}{t}} = \frac{1}{1+t} + \frac{t}{1+t} = 1.$$

Substituting $x \mapsto 4 - x$ shows $\int_0^4 g(x) \, dx = \int_0^4 g(4 - x) \, dx$, so twice the integral is $\int_0^4 1 \, dx = 4$, and the answer is 2. Integrating directly is possible, via $\frac{1}{1+t} = 1 - \frac{t}{1+t}$, but slower and error-prone.

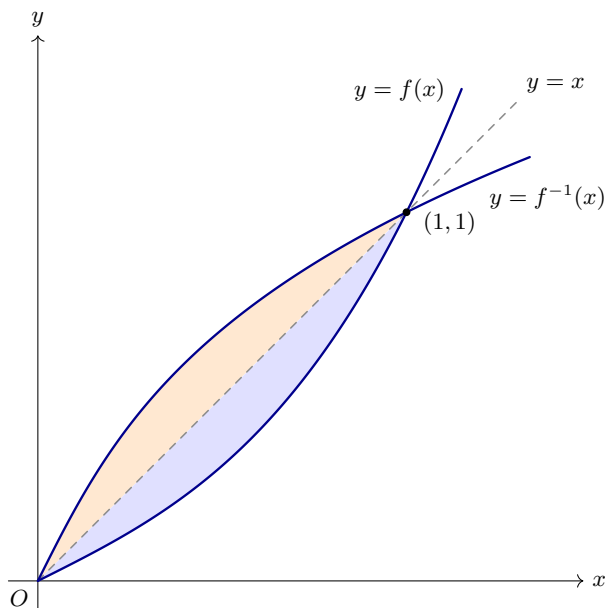
Question 14. Answer **B**.

f is increasing with $f(1) = 4$ and $f(2) = 14$. On $[4, 14]$, the graph of f^{-1} is the reflection of the graph of f on $[1, 2]$, and the two areas fill the L -shaped region between the rectangles $[0, 2] \times [0, 14]$ and $[0, 1] \times [0, 4]$:

$$\int_4^{14} f^{-1}(x) \, dx = 2 \cdot 14 - 1 \cdot 4 - \int_1^2 (x^3 + 3x) \, dx = 24 - \left(\frac{15}{4} + \frac{9}{2}\right) = 24 - \frac{33}{4} = \frac{63}{4}.$$

Option C forgets to subtract the small rectangle $1 \cdot 4$: the formula is $bf(b) - af(a) - \int_a^b f$, and the $af(a)$ term is only zero when $a = 0$.

Question 15. Answer **D**.



f is increasing and odd, and $f(x) = x$ exactly when $x^3 = x$, so the graphs of f and f^{-1} meet the line $y = x$, and each other, at 0 and 1 in the first quadrant. On $(0, 1)$, $f(x) < x$. The region between f and f^{-1} is symmetric about $y = x$, so its area is twice the area between $y = x$ and $y = f(x)$:

$$2 \int_0^1 \left(x - \frac{1}{2}(x^3 + x) \right) dx = \int_0^1 (x - x^3) dx = \frac{1}{4}.$$

No inverse function needs to be found. Option B is the area between f and the line $y = x$ alone.

Question 16. Answer **B**.

The integral $\int_0^1 tf(t) dt$ is just a number; call it B . Then $f(x) = x^2 + Bx$, and substituting back,

$$B = \int_0^1 t(t^2 + Bt) dt = \frac{1}{4} + \frac{B}{3},$$

so $B = \frac{3}{8}$. Hence $\int_0^1 f = \frac{1}{3} + \frac{B}{2} = \frac{1}{3} + \frac{3}{16} = \frac{25}{48}$. Option A comes from stopping at $B = \frac{1}{4}$, having dropped the self-referential term.

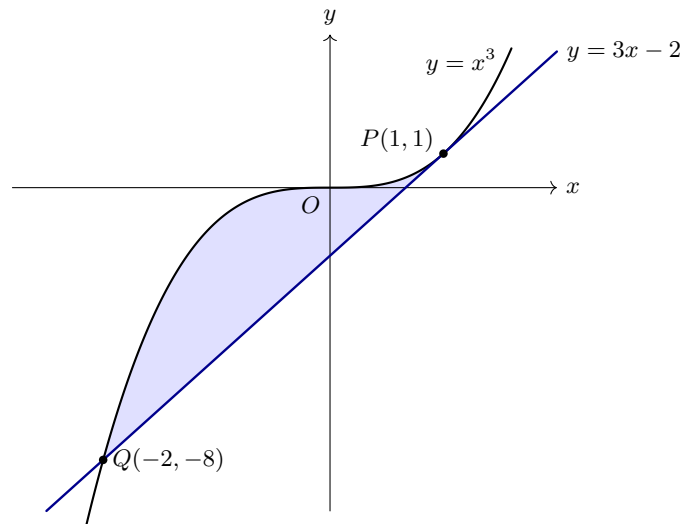
Question 17. Answer **F**.

Both sides are positive, so squaring is reversible: $(|\sin x| + |\cos x|)^2 = 1 + 2|\sin x \cos x| = 1 + |\sin 2x|$. The inequality becomes $1 + |\sin 2x| > \frac{3}{2}$, that is $|\sin 2x| > \frac{1}{2}$. As x runs over $[0, 2\pi]$, $2x$ runs over $[0, 4\pi]$, two full periods, and in each period $|\sin \phi| > \frac{1}{2}$ on a set of length $\frac{4\pi}{3}$. So the set of $2x$ has length $\frac{8\pi}{3}$ and the set of x has length $\frac{4\pi}{3}$. Forgetting to halve at the end gives $\frac{8\pi}{3}$; forgetting that there are two periods gives option C.

Question 18. Answer **E**.

With $s = \sin x$ and $\cos 2x = 1 - 2s^2$, the condition is $g(s) = 2s^2 - as + a \geq 0$ for all $s \in [-1, 1]$. The minimum of g on $[-1, 1]$ depends on where the vertex $s = \frac{a}{4}$ lies. If $-4 \leq a \leq 4$, the minimum is $g(\frac{a}{4}) = a - \frac{a^2}{8}$, which is non-negative for $0 \leq a \leq 8$, so here $0 \leq a \leq 4$. If $a > 4$, the vertex is to the right of the interval and the minimum is $g(1) = 2 > 0$, so every $a > 4$ works. If $a < -4$, the minimum is $g(-1) = 2 + 2a < 0$. Altogether $a \geq 0$. Option B is the discriminant condition applied without restricting s to $[-1, 1]$: it wrongly throws away every $a > 8$.

Question 19. Answer **C**.



The tangent at P is $y = 3x - 2$, and $x^3 - (3x - 2) = x^3 - 3x + 2 = (x - 1)^2(x + 2)$, where the double factor is forced by tangency. So Q has $x = -2$, and the curve lies above the tangent on $(-2, 1)$. With $u = x - 1$,

$$\int_{-2}^1 (x - 1)^2(x + 2) dx = \int_{-3}^0 (u^3 + 3u^2) du = 0 - \left(\frac{81}{4} - 27\right) = \frac{27}{4}.$$

Knowing that the factorisation must contain $(x - 1)^2$ saves the polynomial division, and the shift $u = x - 1$ saves expanding.

Question 20. Answer **B**.

Factor out $\frac{1}{n}$: the sum is $\frac{1}{n} \sum_{k=1}^n \frac{1}{(1+k/n)^2}$, a Riemann sum for $\int_0^1 \frac{dx}{(1+x)^2}$. As $n \rightarrow \infty$ it tends to

$$\left[-\frac{1}{1+x}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

The recognition is the question: once the sum is seen as rectangles of width $\frac{1}{n}$ under a curve, the limit is an area.