

HU Collection

ESAT Mock Set D: Mathematics 2

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	B	D	B	B	C	C	E	A	F	A

Q	11	12	13	14	15	16	17	18	19	20
Ans	F	D	C	C	A	A	E	F	E	D

Q	21	22	23	24	25	26	27			
Ans	A	B	F	E	D	B	C			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer B.

Each transformation acts on x in the current equation. The translation gives $y = \sqrt{x-3}$. A stretch parallel to the x -axis with scale factor $\frac{1}{2}$ replaces x by $2x$, giving $y = \sqrt{2x-3}$. The reflection in the y -axis replaces x by $-x$, giving $g(x) = \sqrt{-2x-3}$, so $g(-5) = \sqrt{7}$.

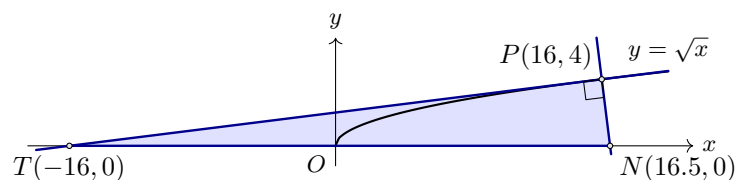
Applying the stretch as if it acted on the whole bracket $x-3$ gives $\sqrt{-2x-6}$ and the answer 2; translating the wrong way gives $\sqrt{-2x+3}$ and $\sqrt{13}$; replacing x by $\frac{x}{2}$ for the stretch gives $\sqrt{-\frac{x}{2}-3}$, which is undefined at -5 .

Question 2. Answer D.

The line is $\log_2 y = 3\log_4 x + 2$. Since $\log_4 x = \frac{1}{2}\log_2 x$, this is $\log_2 y = \frac{3}{2}\log_2 x + 2$, so $y = 4x^{3/2}$. Then $y = x$ gives $4x^{1/2} = 1$, so $x = \frac{1}{16}$.

The trap is to treat the horizontal axis as $\log_2 x$, which gives $y = 4x^3$ and the answer $\frac{1}{2}$.

Question 3. Answer B.



Here $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{8}$ at P . The tangent $y - 4 = \frac{1}{8}(x - 16)$ meets $y = 0$ at $x = -16$. The normal has gradient -8 , so $y - 4 = -8(x - 16)$ meets $y = 0$ at $x = 16.5$. The base TN is 32.5 and the height is 4, so the area is 65.

Using $-\frac{1}{8}$ as the normal gradient (forgetting the reciprocal) puts N at $x = 48$ and gives 128; forgetting the half gives 130.

Question 4. Answer B.

$P(1) = -1 + 9 - 23 + 15 = 0$, so $x - 1$ is a factor, and division gives $P(x) = -(x - 1)(x^2 - 8x + 15) = -(x - 1)(x - 3)(x - 5)$. So $P(x) > 0$ exactly when $(x - 1)(x - 3)(x - 5) < 0$, which is $x < 1$ or $3 < x < 5$. With $x > 0$ the answer is $0 < x < 1$ or $3 < x < 5$.

Forgetting the leading minus sign reverses every region and gives $1 < x < 3$ or $x > 5$; ignoring the small region near the origin gives $3 < x < 5$ only.

Question 5. Answer **C**.

Write $c = \cos 2x$ (non-zero, as $\tan 2x$ must be defined). Then $\frac{\sin^2 2x}{\cos 2x} = \frac{3}{2}$ gives $2(1 - c^2) = 3c$, so $2c^2 + 3c - 2 = 0$ and $(2c - 1)(c + 2) = 0$. Only $c = \frac{1}{2}$ is possible. As x runs over $[0, 2\pi]$, $2x$ runs over $[0, 4\pi]$, giving $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$, so $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$, with sum 4π .

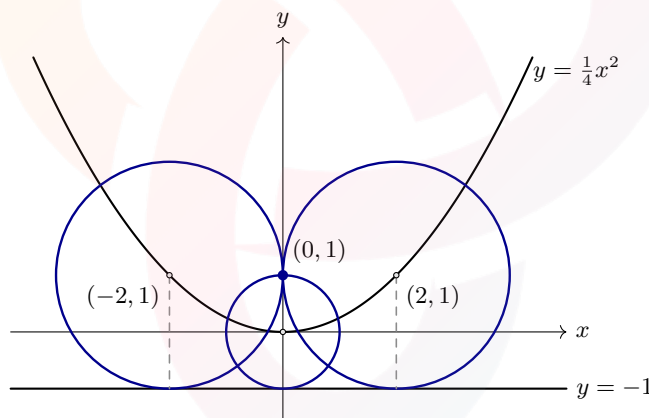
Working only over $0 \leq 2x \leq 2\pi$ gives 2π ; summing the values of $2x$ instead of x gives 8π .

Question 6. Answer **C**.

The left side is never negative, so we need $3x > 0$, that is $x > 0$. Then $|x^3 - 4x| = x|x^2 - 4|$, and dividing by $x > 0$ gives $|x^2 - 4| < 3$, so $1 < x^2 < 7$, that is $1 < x < \sqrt{7}$.

At $x = 2$ the left side is $0 < 6$, so 2 is included. Dividing by x without first noting $x > 0$ produces the spurious negative interval.

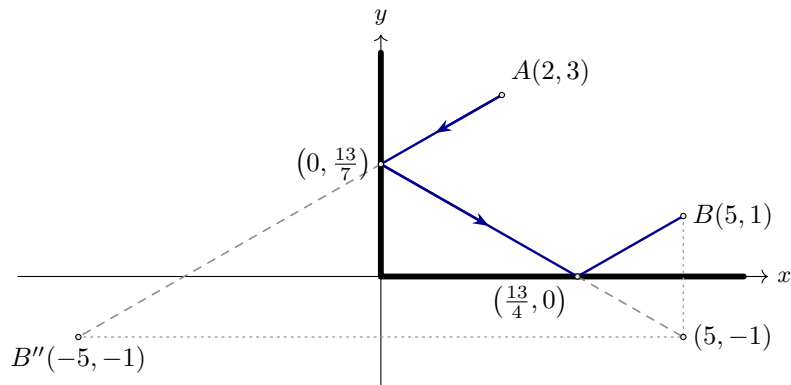
Question 7. Answer **E**.



A circle with centre $(t, \frac{1}{4}t^2)$ touching $y = -1$ has radius $\frac{1}{4}t^2 + 1$. The squared distance from the centre to $(0, 1)$ is $t^2 + (\frac{1}{4}t^2 - 1)^2 = (\frac{1}{4}t^2 + 1)^2$, so $(0, 1)$ is on every circle: it is the focus of the parabola, whose directrix is $y = -1$.

The circles with $t = 0$ and $t = \pm 2$ already have $(0, 1)$ as their only common point. The point $(0, \frac{1}{4})$ is the focus of $y = x^2$, not of this parabola.

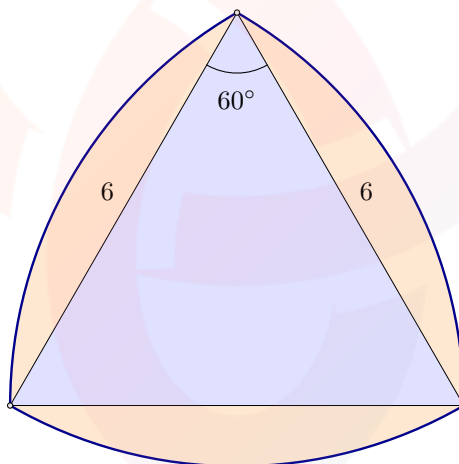
Question 8. Answer **A**.



Unfold the path. Reflect B in the x -axis (the last cushion) to $(5, -1)$, then reflect that in the y -axis to $B''(-5, -1)$. The ball's path has the same length as the straight segment AB'' , which is $\sqrt{7^2 + 4^2} = \sqrt{65}$. Checking: AB'' meets the y -axis at $(0, \frac{13}{7})$ and the next leg meets the x -axis at $(\frac{13}{4}, 0)$, both on the positive half-axes, so the path is genuine.

Reflecting in only one cushion gives 5 or $\sqrt{53}$.

Question 9. Answer **F**.



Each arc has radius 6 and subtends $\frac{\pi}{3}$ at its centre, so each sector has area $\frac{1}{2} \cdot 36 \cdot \frac{\pi}{3} = 6\pi$, and each circular segment has area $6\pi - 9\sqrt{3}$ (the triangle has area $\frac{\sqrt{3}}{4} \cdot 36 = 9\sqrt{3}$). The region is the triangle plus three segments: $9\sqrt{3} + 3(6\pi - 9\sqrt{3}) = 18\pi - 18\sqrt{3}$.

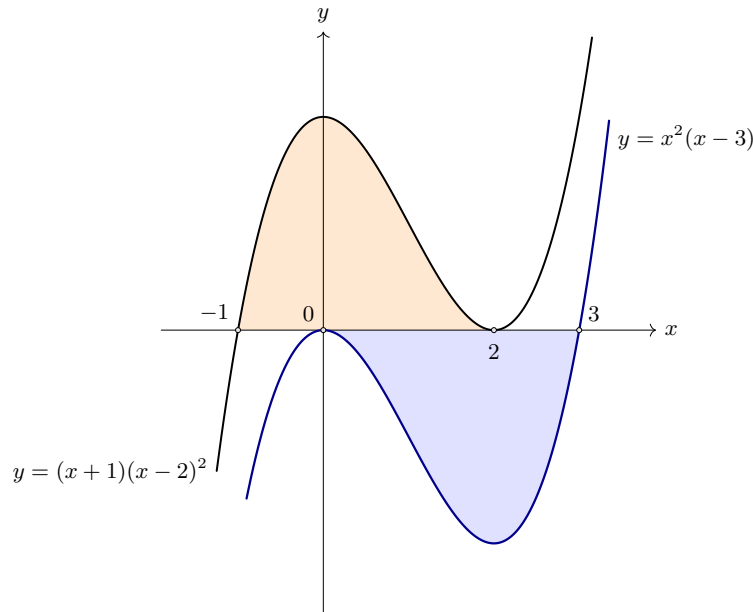
Adding three whole sectors to the triangle gives $18\pi + 9\sqrt{3}$, counting the triangle four times; the three segments alone give $18\pi - 27\sqrt{3}$.

Question 10. Answer **A**.

The remainder $r(x) = ax + b$ agrees with $p(x)^3$ at the roots $x = \pm 1$ of $x^2 - 1$. From the given remainder, $p(1) = 4$ and $p(-1) = 2$, so $p(1)^3 = 64$ and $p(-1)^3 = 8$. Then $a + b = 64$ and $-a + b = 8$, so $b = 36$ and $a = 28$.

Equivalently, $(x+3)^3 = x^3 + 9x^2 + 27x + 27$ and replacing x^2 by 1 gives $x + 9 + 27x + 27 = 28x + 36$. Leaving the cube unreduced is not a remainder, since its degree is too large.

Question 11. Answer **F**.



Integrating, $y = x^3 - 3x^2 + c$. The x -axis is a tangent where $y = 0$ and $\frac{dy}{dx} = 0$, so at $x = 0$ (giving $c = 0$) or $x = 2$ (giving $c = 4$). There are two possible curves, $y = x^2(x - 3)$ and $y = (x + 1)(x - 2)^2$. For the first, the region is $0 \leq x \leq 3$ with area $\left| \int_0^3 (x^3 - 3x^2) dx \right| = \left| \frac{81}{4} - 27 \right| = \frac{27}{4}$. For the second, the region is $-1 \leq x \leq 2$ with area $\int_{-1}^2 (x^3 - 3x^2 + 4) dx = \frac{27}{4}$ as well, so the answer is the same either way. Adding the two cases gives the trap $\frac{27}{2}$; integrating the second curve from 0 to 2 gives 4.

Question 12. Answer **D**.

The moves go east 2, north $\frac{1}{2}$, west $\frac{1}{2}$, south $\frac{1}{4}$, east $\frac{1}{8}$, and so on. The x -coordinate is $2 - \frac{1}{2} + \frac{1}{8} - \dots$, a geometric series with ratio $-\frac{1}{4}$, so it tends to $\frac{2}{1 + \frac{1}{4}} = \frac{8}{5}$. Likewise $y = 1 - \frac{1}{4} + \frac{1}{16} - \dots = \frac{4}{5}$. The

distance is $\sqrt{\frac{64}{25} + \frac{16}{25}} = \frac{\sqrt{80}}{5} = \frac{4\sqrt{5}}{5}$.

The total length travelled is 4, which is the trap; $\frac{12}{5}$ adds the coordinates instead of using Pythagoras.

Question 13. Answer **C**.

The polygon splits into n isosceles triangles with two sides 1 and apex angle $\frac{2\pi}{n}$, so its area is $\frac{n}{2} \sin \frac{2\pi}{n}$. The areas increase with n (each polygon gets closer to filling the circle), and for $n = 12$ the area is exactly $6 \sin \frac{\pi}{6} = 3$. So the first polygon with area greater than 3 has 13 sides.

No estimate of π is needed; the trap is to stop at 12, where the area equals 3 but does not exceed it.

Question 14. Answer **C**.

Track $L = \log_2(\text{display})$, which starts at 1. Button S doubles L and button H subtracts 1. Work backwards from $L = 100$: an even value may come from half of it (by S) or from one more (by H), and an odd value can only come from one more. The efficient route is

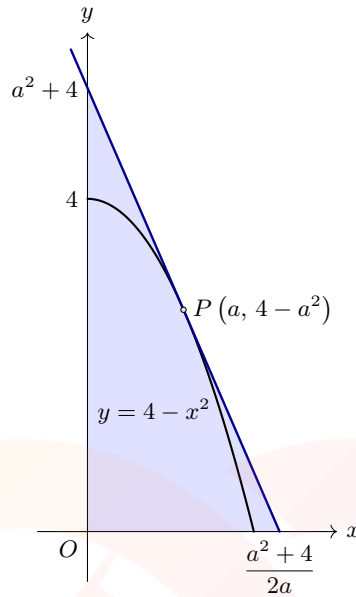
$$1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \rightarrow 7 \rightarrow 14 \rightarrow 13 \rightarrow 26 \rightarrow 25 \rightarrow 50 \rightarrow 100,$$

which takes 10 presses.

Fewer is impossible. Six doublings reach at most $2^6 = 64$, so at least seven are needed. A subtraction made when j doublings remain lowers the final value of L by 2^j , so with seven doublings the subtractions must account for $128 - 100 = 28 = 16 + 8 + 4$, which needs at least three of them (three 1s in binary): $7 + 3 = 10$. More doublings only make things worse, since $256 - 100 = 156$

already needs four. An increment button would allow 8 presses via $100 = 1100100_2$, but there is no increment here.

Question 15. Answer **A**.



At $P(a, 4 - a^2)$ with $0 < a < 2$, the tangent is $y = -2ax + a^2 + 4$. Its intercepts are $a^2 + 4$ on the y -axis and $\frac{a^2 + 4}{2a}$ on the x -axis, so the area is

$$A = \frac{(a^2 + 4)^2}{4a} = \frac{a^3}{4} + 2a + \frac{4}{a}.$$

Then $\frac{dA}{da} = \frac{3}{4}a^2 + 2 - \frac{4}{a^2} = 0$ gives $3a^4 + 8a^2 - 16 = 0$, so $(3a^2 - 4)(a^2 + 4) = 0$ and $a^2 = \frac{4}{3}$. This gives $A = \frac{(16/3)^2}{8/\sqrt{3}} = \frac{32\sqrt{3}}{9}$.

Expanding into powers of a avoids the quotient rule. The tempting $a = 1$ gives $\frac{25}{4}$, which is only slightly bigger.

Question 16. Answer **A**.

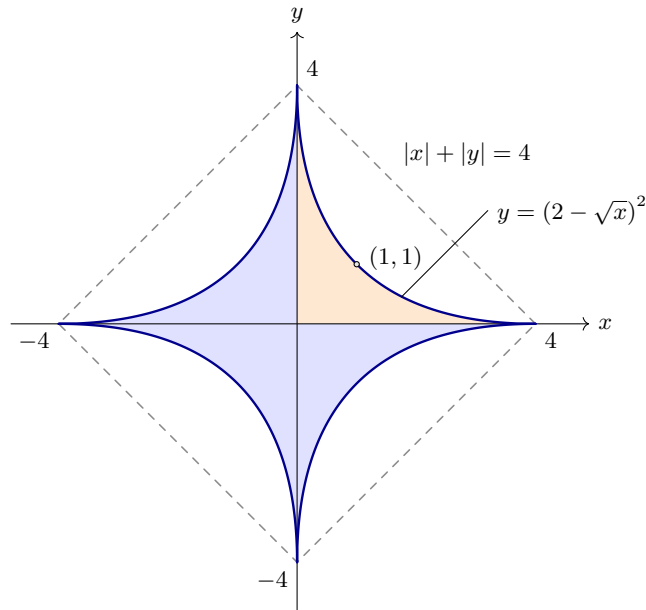
The left side $f(x)$ is defined for $-3 \leq x \leq 5$ and is positive, and

$$f(x)^2 = 8 + 2\sqrt{(x+3)(5-x)} = 8 + 2\sqrt{16 - (x-1)^2}.$$

So f is symmetric about $x = 1$, rises from $f(-3) = 2\sqrt{2}$ to its maximum $f(1) = 4$, then falls back to $f(5) = 2\sqrt{2}$. Each value of c with $2\sqrt{2} \leq c < 4$ is taken exactly twice (once on each side of $x = 1$), $c = 4$ once, and other values not at all.

The value $c = 2\sqrt{2}$ is included: it is attained at both endpoints $x = -3$ and $x = 5$.

Question 17. Answer **E**.



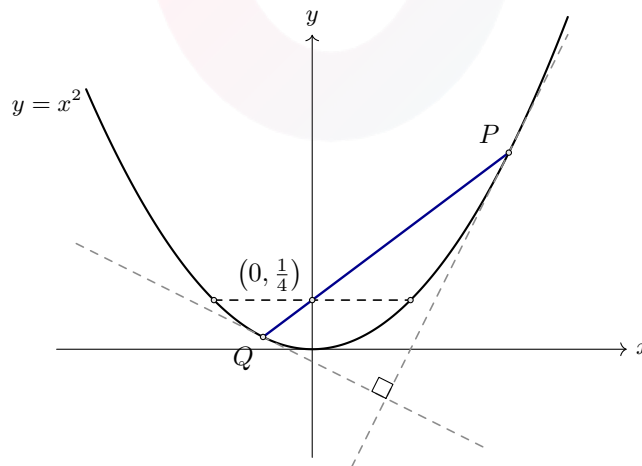
The region is symmetric in both axes, so find the part in the first quadrant and multiply by 4. There, $\sqrt{y} \leq 2 - \sqrt{x}$ with $0 \leq x \leq 4$, that is $y \leq (2 - \sqrt{x})^2 = 4 - 4x^{1/2} + x$. The area is

$$\int_0^4 (4 - 4x^{1/2} + x) dx = 16 - \frac{8}{3} \cdot 8 + 8 = \frac{8}{3},$$

so the total is $\frac{32}{3}$.

The boundary curves inwards (the point $(1, 1)$ is on it), so the area is less than that of the square $|x| + |y| \leq 4$, which is 32. Forgetting the factor 4 gives $\frac{8}{3}$.

Question 18. Answer **F**.



In the figure, the dashed chord is the shortest one, with $p = 1/2$ and $q = -1/2$.

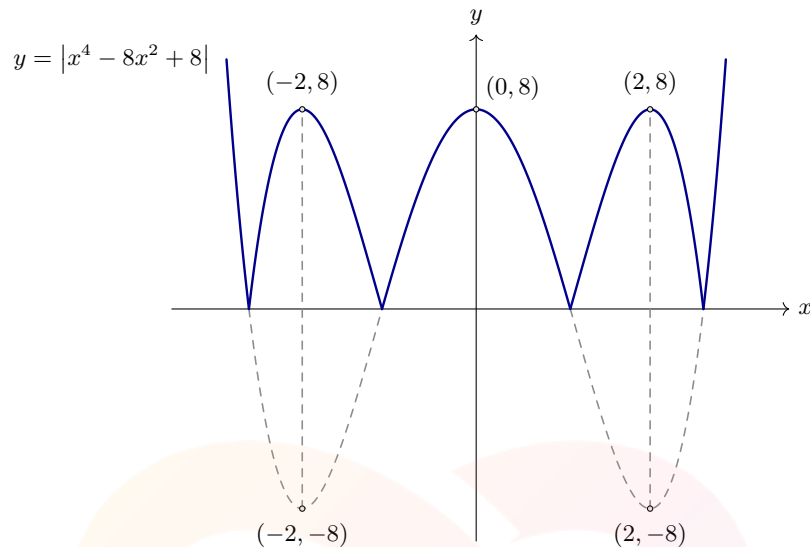
Let $P = (p, p^2)$ and $Q = (q, q^2)$. The tangent gradients are $2p$ and $2q$, so $4pq = -1$. Then

$$PQ^2 = (p - q)^2 (1 + (p + q)^2).$$

Writing $s = p + q$, we have $(p - q)^2 = s^2 - 4pq = s^2 + 1$, so $PQ^2 = (1 + s^2)^2$ and $PQ = 1 + s^2 \geq 1$, with equality when $p = \frac{1}{2}$, $q = -\frac{1}{2}$.

This is the chord through the focus $(0, \frac{1}{4})$ perpendicular to the axis: every such chord passes through the focus.

Question 19. Answer **E**.



The figure shows the case $k = -8$, which lies in the range found.

Let $g(x) = x^4 - 8x^2 - k$. Since $g'(x) = 4x(x^2 - 4)$, g has a local maximum $g(0) = -k$ and local minima $g(\pm 2) = -16 - k$. Taking the modulus keeps a local maximum of g where $g > 0$, turns a local minimum of g where $g < 0$ into a local maximum, and creates only minima (corners) where g crosses zero.

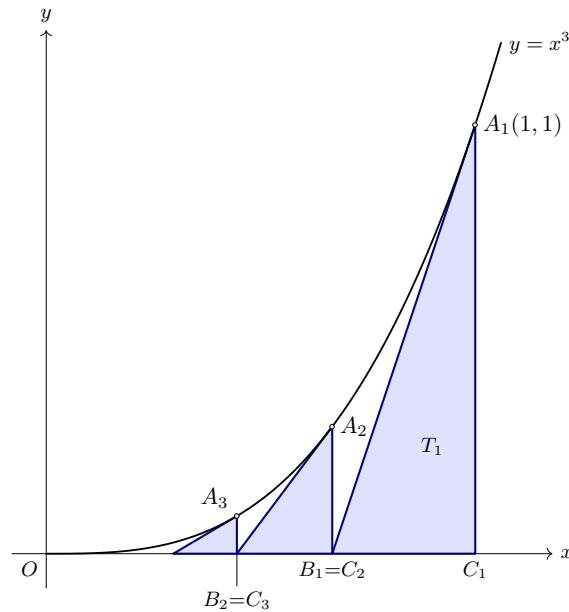
So $x = 0$ gives a local maximum of $|g|$ exactly when $-k > 0$, and $x = \pm 2$ give two exactly when $-16 - k < 0$. Three maxima need $-16 < k < 0$, giving the 15 integers $-15, \dots, -1$. At $k = 0$ the point $x = 0$ becomes a minimum of $|g|$ (only 2 maxima), and at $k = -16$ the minima touch the axis (only 1).

Question 20. Answer **D**.

A repeated factor $(x - 1)^2$ of $p(x) - 1$ means $p(1) = 1$ and $p'(1) = 0$; similarly $p(-1) = -1$ and $p'(-1) = 0$. So the quadratic $p'(x)$ has roots ± 1 : $p'(x) = k(1 - x^2)$ and $p(x) = k\left(x - \frac{x^3}{3}\right) + d$. The conditions $p(1) = 1$, $p(-1) = -1$ give $d = 0$ and $\frac{2k}{3} = 1$, so $p(x) = \frac{1}{2}(3x - x^3)$ and $p(3) = \frac{1}{2}(9 - 27) = -9$.

A sign slip in k gives $\frac{1}{2}(x^3 - 3x)$ and the answer 9, but that polynomial has $p(1) = -1$.

Question 21. Answer **A**.



At $A = (a, a^3)$ the tangent $y = a^3 + 3a^2(x - a)$ meets $y = 0$ at $x = a - \frac{a}{3} = \frac{2a}{3}$. So the x -coordinates are $1, \frac{2}{3}, (\frac{2}{3})^2, \dots$, and the triangle at $A = (a, a^3)$ has base $\frac{a}{3}$ and height a^3 , with area $\frac{a^4}{6}$.

The areas form a geometric series with first term $\frac{1}{6}$ and ratio $(\frac{2}{3})^4 = \frac{16}{81}$, so the total is $\frac{1/6}{1 - 16/81} = \frac{1}{6} \cdot \frac{81}{65} = \frac{27}{130}$. As a check, this is less than $\int_0^1 x^3 dx = \frac{1}{4}$, since the triangles lie under the curve. Using ratio $\frac{2}{3}$ or $\frac{4}{9}$ gives $\frac{1}{2}$ or $\frac{3}{10}$.

Question 22. Answer **B**.

If θ is the angle between the hands, the cosine rule gives the distance d with $d^2 = 34 - 30 \cos \theta$, so $2 \leq d \leq 8$. The minute hand gains 11 full turns on the hour hand in 12 hours, so θ goes through 11 complete cycles.

In each cycle, d takes each value strictly between 2 and 8 twice and the values 2 and 8 once each. The whole numbers 3, 4, 5, 6, 7 therefore occur 22 times each, and 2 and 8 occur 11 times each (the hands coincide at noon but not at midnight within the period). The total is $5 \times 22 + 2 \times 11 = 132$. Counting every whole number 22 times gives 154; leaving out 2 and 8 gives 110.

Question 23. Answer **F**.

The final output at $x = 0$ is b , and it can be read off a difference table. The values of $p(x) = (x(x+1))^3$ at $x = 0, 1, 2, 3, 4, 5$ are 0, 8, 216, 1728, 8000, 27000. Taking differences five times:

$$\begin{array}{ccccccccc} 8, & 208, & 1512, & 6272, & 19000; & 200, & 1304, & 4760, & 12728; \\ & & & & & 1104, & 3456, & 7968; & 2352, & 4512; & 2160. \end{array}$$

So $b = 2160$.

Alternatively, the machine lowers the degree by one, sends x^6 to a polynomial that becomes $720x + 1800$ after five uses, and sends $3x^5$ to the constant $3 \times 120 = 360$ after five uses, while $3x^4 + x^3$ disappears; again $b = 1800 + 360 = 2160$. The leading term alone gives 1800; $a = 720$ is the trap of reporting the wrong coefficient.

Question 24. Answer **E**.

Since $f'(x) = 3x^2 - 21$, f decreases on $[0, \sqrt{7}]$ and increases for $x \geq \sqrt{7}$. If $a > \sqrt{7}$ then f increases on the whole interval and $M(a) = f(a+3) > f(\sqrt{7}+3)$, which is larger than values found below. For $0 \leq a \leq \sqrt{7}$ the interval contains $\sqrt{7}$, so the greatest value is at an end: $M(a) = \max(f(a), f(a+3))$.

As a increases, $f(a)$ falls and $f(a + 3)$ rises, so the larger of the two is least where they are equal. Now $f(a + 3) - f(a) = 9a^2 + 27a - 36 = 9(a - 1)(a + 4)$, so they are equal at $a = 1$, and $M(1) = f(1) = f(4) = -20$.

The trap is the minimum value of f itself, $f(\sqrt{7}) = -14\sqrt{7}$: that is the least value on the interval, not the greatest. Taking $a = 0$ and reading off $f(3) = -36$ makes the same mistake.

Question 25. Answer D.

Doubling a number with d digits gives a number with d or $d + 1$ digits, and in the second case the new number lies between 10^d and 2×10^d , so it starts with 1. Conversely, if 2^n lies between 10^d and 2×10^d , then 2^{n-1} lies between $5 \times 10^{d-1}$ and 10^d , so it has one digit fewer. Hence among $2^2, \dots, 2^{100}$ the powers starting with 1 are exactly those where the digit count increases.

Since $100 \log_{10} 2 = 30.103$, 2^{100} has 31 digits, while 2^1 has 1 digit. The digit count rises 30 times, so the answer is 30. Benford's law suggests about 30%, but the exact count needs this argument; 31 is the digit count of 2^{100} .

Question 26. Answer B.

First treat one plate. Light gets through by crossing both faces, possibly after an even number of internal reflections, so the fraction transmitted is

$$T = \left(\frac{2}{3}\right)^2 \left(1 + \frac{1}{9} + \frac{1}{81} + \dots\right) = \frac{4}{9} \cdot \frac{9}{8} = \frac{1}{2},$$

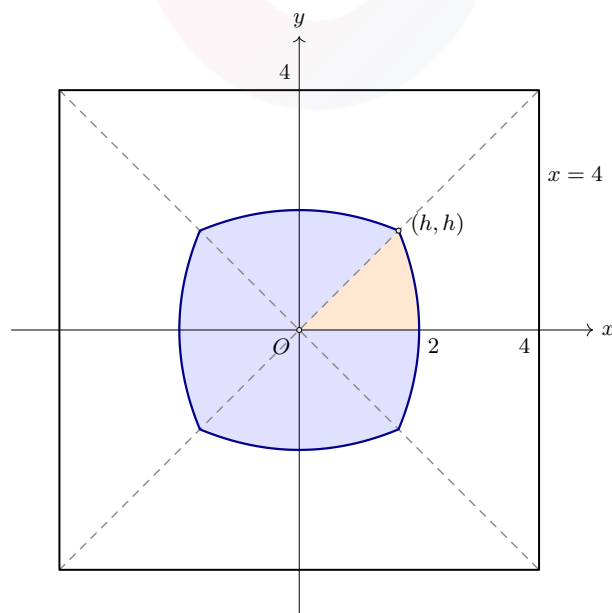
and since nothing is absorbed, the plate reflects $R = \frac{1}{2}$ overall (directly, $\frac{1}{3} + \frac{4}{9} \cdot \frac{1}{3} \cdot \frac{9}{8} = \frac{1}{2}$).

Now treat each plate as a single unit with $T = R = \frac{1}{2}$. Light passes the first plate, then either passes the second plate or is reflected back and forth between the plates:

$$\frac{1}{2} \cdot \frac{1}{2} \left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right) = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}.$$

Ignoring the repeated reflections gives $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$, or $\frac{1}{4}$ at plate level. Using the single-face value $\frac{1}{3}$ for reflection between the plates gives $\frac{9}{32}$.

Question 27. Answer C.



By symmetry, R is made of eight congruent pieces, one in each of the triangles cut off by the axes and the diagonals. In the triangle $0 \leq y \leq x$ the nearest side is $x = 4$, so the condition is

$x^2 + y^2 < (4 - x)^2$, that is $x < 2 - \frac{1}{8}y^2$. The boundary is the parabola with focus at the centre and directrix $x = 4$.

The parabola meets the diagonal $y = x$ where $x^2 + 8x - 16 = 0$, so at $h = 4\sqrt{2} - 4$. The piece is $0 \leq y \leq h$, $y \leq x < 2 - \frac{1}{8}y^2$, with area

$$\int_0^h \left(2 - y - \frac{1}{8}y^2\right) dy = 2h - \frac{1}{2}h^2 - \frac{1}{24}h^3.$$

With $h^2 = 48 - 32\sqrt{2}$ and $h^3 = 320\sqrt{2} - 448$, this is $\frac{32\sqrt{2} - 40}{3} = \frac{8(4\sqrt{2} - 5)}{3}$. Hence the area of R is $\frac{64(4\sqrt{2} - 5)}{3} \approx 14.0$.

Using only the two sides $x = \pm 4$ gives the region between two parabolas, of area $\frac{64}{3}$. Forgetting to multiply by 8 gives $\frac{8(4\sqrt{2} - 5)}{3}$, and multiplying by 4 instead gives $\frac{32(4\sqrt{2} - 5)}{3}$.

