

HU Collection ESAT Mock Set D: Mathematics 1

Worked Solutions

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Answer key

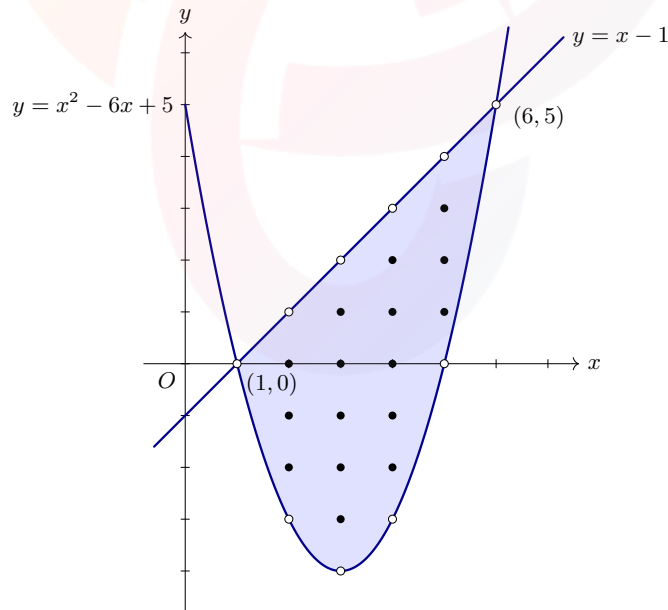
Q	1	2	3	4	5	6	7	8	9	10
Ans	D	A	D	B	C	A	A	F	B	B

Q	11	12	13	14	15	16	17	18	19	20
Ans	F	C	D	F	E	E	B	D	C	C

Q	21	22	23	24	25	26	27			
Ans	E	C	B	E	A	A	F			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer **D**.



In the figure, the filled dots are the points counted and the hollow dots lie on the boundary.

The graphs meet where $x^2 - 6x + 5 = x - 1$, that is $x^2 - 7x + 6 = 0$, so at $x = 1$ and $x = 6$; between these the line lies above the parabola. An interior point needs $2 \leq x \leq 5$ and $x^2 - 6x + 5 < y < x - 1$.

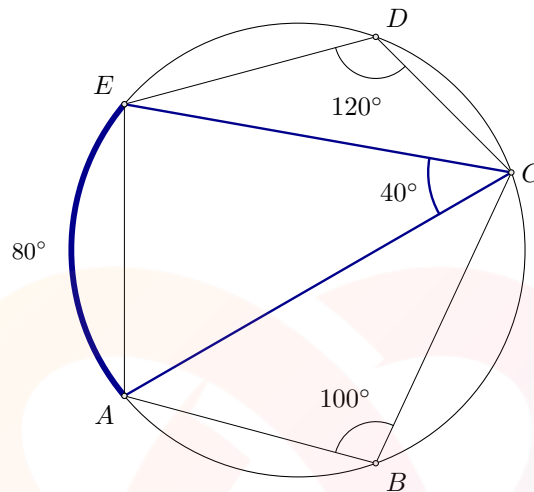
For $x = 2$: $-3 < y < 1$, giving 3 values. For $x = 3$: $-4 < y < 2$, giving 5. For $x = 4$: $-3 < y < 3$, giving 5. For $x = 5$: $0 < y < 4$, giving 3. The total is 16. Including the points on the boundary would give 26.

Question 2. Answer **A**.

Car A uses $6 \times 4.8 = 28.8$ litres. For car B , 45 miles per gallon is $45 \times 1.6 = 72$ km per 4.5 litres, which is 16 km per litre, so it uses $\frac{480}{16} = 30$ litres. Car A uses less, by 1.2 litres.

The two measures run in opposite directions: litres per 100 km is fuel per distance, while miles per gallon is distance per fuel, so a larger number is worse for A and better for B . Converting the wrong way (dividing by 1.6 instead of multiplying) makes car B look far thirstier and gives the 48 litre option.

Question 3. Answer **D**.



An inscribed angle is half of the arc it stands on, the arc not containing its vertex. $\angle ABC = 100^\circ$ stands on the arc $AEDC$, so that arc is 200° and the arc ABC is 160° . $\angle CDE = 120^\circ$ stands on the arc $CBAE$, which is therefore 240° , so the arc CDE is 120° .

The remaining arc EA , containing none of B, C, D , is $360^\circ - 160^\circ - 120^\circ = 80^\circ$. The angle $\angle ACE$ stands on this arc, so $\angle ACE = 40^\circ$. The answer 80° forgets to halve the arc.

Question 4. Answer **B**.

Let the total of the marks be S , the lowest mark L and the highest H . Then

$$S - L = 6 \left(\frac{S}{7} + 3 \right) \quad \text{and} \quad S - H = 6 \left(\frac{S}{7} - 4 \right).$$

Subtracting, $H - L = 6(3 + 4) = 42$. The total S is never needed. The multiplier is the number of marks that remain, 6; using 7 gives the trap 49.

Such marks do exist: 10, 25, 26, 27, 28, 28, 52 have mean 28, and the two reduced means are 31 and 24.

Question 5. Answer **C**.

Write $N = 1000k + m$ with $0 \leq m \leq 999$. Rounding directly gives $1000k$ exactly when $m \leq 499$, and $1000(k + 1)$ otherwise. In the chain, m first goes to the nearest 10. If $m \geq 445$, this gives at least 450, the next step gives 500 or more, and the last step gives $1000(k + 1)$. If $m \leq 444$, the chain gives at most 440, then at most 400, and finally $1000k$.

So the two methods differ exactly when $445 \leq m \leq 499$: 55 values in each block of 1000. From 1 to 10 000 there are 10 such blocks ($k = 0$ to 9; the number 10 000 itself causes no difference), giving $10 \times 55 = 550$. Forgetting that 445 to 449 round up to 450 and then to 500 gives 500.

Question 6. Answer **A**.

Each hour the volume is multiplied by $0.729 = 0.9^3$. The balloon stays spherical, so all its shapes are similar: the radius is multiplied by 0.9 each hour and the surface area by $0.9^2 = 0.81$.

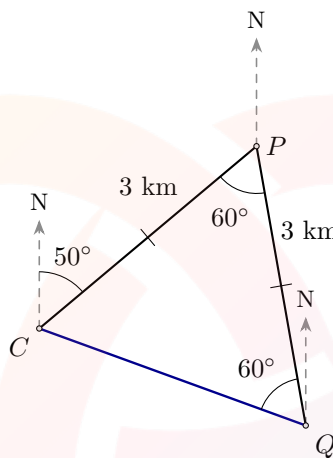
After 2 hours the surface area has been multiplied by $0.81^2 = 0.6561$, which is a decrease of 34.39%. Adding the hourly decreases ($2 \times 19\% = 38\%$) ignores compounding, 46.86% is the decrease in volume rather than area, and 18.07% comes from wrongly taking two thirds of the volume percentage.

Question 7. Answer A.

Completing the square, a curve of this form with turning point (h, h) is $y = (x - h)^2 + h$. Passing through $(4, 10)$ requires $(4 - h)^2 + h = 10$, so $h^2 - 7h + 6 = 0$ and $h = 1$ or $h = 6$.

Expanding, $c = h^2 + h$, which is 2 or 42. The sum is 44. The value 7 is the sum of the possible values of h , and -14 is the sum of the possible values of b .

Question 8. Answer F.



At P , the direction back to C is on a bearing of $050^\circ + 180^\circ = 230^\circ$ and the direction on to Q is 170° , so $\angle CPQ = 230^\circ - 170^\circ = 60^\circ$. Since $PC = PQ = 3$ km, triangle CPQ is isosceles with a 60° apex angle, so it is equilateral and $QC = 3$ km $= 300\,000$ cm, which is $\frac{300\,000}{25\,000} = 12$ cm on the map.

At Q , the bearing back to P is $170^\circ + 180^\circ = 350^\circ$, and C is reached by turning a further 60° anticlockwise (the triangle lies to the west of PQ , since C is west of Q), giving $350^\circ - 60^\circ = 290^\circ$. The bearing 110° is that of Q from C , the reverse direction.

Question 9. Answer B.

Since $36 = 2^2 \times 3^2$, the only usable digits are 1, 2, 3, 4, 6, 9. List the multisets of four digits with product 36 systematically by the largest digit:

$$\{1, 1, 4, 9\}, \{1, 2, 2, 9\}, \{1, 1, 6, 6\}, \{1, 2, 3, 6\}, \{1, 3, 3, 4\}, \{2, 2, 3, 3\}.$$

Their numbers of different orders are 12, 12, 6, 24, 12, 6, which total 72.

The usual slips are missing $\{2, 2, 3, 3\}$ (the only one with no 1) or $\{1, 1, 6, 6\}$, either of which gives 66.

Question 10. Answer B.

After the first pour, A holds 400 ml containing 80 ml of concentrate, and B holds 600 ml containing $200 + 40 = 240$ ml, so B is now 40% concentrate. Pouring back v ml gives

$$\frac{80 + 0.4v}{400 + v} = 0.3,$$

so $80 + 0.4v = 120 + 0.3v$ and $v = 400$.

Using 50% for B , forgetting that it was diluted, gives 200 ml; forgetting that A lost 200 ml gives 600 ml.

Question 11. Answer **F**.

Let the buses travel at speed V , the cyclist at speed u , and let the buses leave every T minutes, so that buses travelling in the same direction are a distance $d = VT$ apart. Oncoming buses close on her at $V + u$ and overtaking buses at $V - u$, so

$$\frac{d}{V + u} = 6 \quad \text{and} \quad \frac{d}{V - u} = 12.$$

Hence $V + u = \frac{d}{6}$ and $V - u = \frac{d}{12}$; adding, $2V = \frac{d}{4}$, so $V = \frac{d}{8}$ and $T = \frac{d}{V} = 8$ minutes.

The interval is the harmonic mean of 6 and 12, not their ordinary mean 9. (The cyclist rides at a third of the buses' speed.)

Question 12. Answer **C**.

I is true: the mean of Z is a weighted average of the two means. III is true: Z contains the smallest and largest values of both sets.

II is true. Say $m_X \leq m_Y$. At least half of X and at least half of Y lie at or below m_Y , so at least half of Z does; likewise at least half of Z lies at or above m_X . If Z has an odd number of values, this traps its middle value between m_X and m_Y . If it has an even number, the only delicate case is when exactly half of Z lies at or below m_Y ; then the two middle values of Z are at most the two middle values of Y , or at most the two middle values of X , so their mean is at most m_Y . The lower bound is similar.

IV is false: $X = \{1, 1, 2, 3\}$ and $Y = \{3, 4, 5, 5\}$ have modes 1 and 5, but in Z the values 1, 3 and 5 each occur twice, so 3 is also a mode of Z .

Question 13. Answer **D**.

Let the airspeed be v and the one-way distance d . Then

$$\frac{d}{v + 10} + \frac{d}{v - 10} = 1.25 \times \frac{2d}{v}.$$

The left side is $\frac{2dv}{v^2 - 100}$, so $v^2 = 1.25(v^2 - 100)$, giving $0.25v^2 = 125$, $v^2 = 500$ and $v = 10\sqrt{5} \approx 22.4$.

The wind does not cancel out over the round trip: the drone spends longer on the slow leg, so the head wind costs more time than the tail wind saves.

Question 14. Answer **F**.

Frequency is frequency density times class width, so the first two classes contain $2 \times 10 = 20$ and $6 \times 5 = 30$ customers, and the last contains $2.5(b - 15)$. The median estimate 20 lies 5 minutes into the last class, so half of the total is $20 + 30 + 2.5 \times 5 = 62.5$. The total is 125, the last class contains 75 customers, its width is $\frac{75}{2.5} = 30$, so $b = 45$ and its midpoint is 30.

The estimated mean is

$$\frac{20 \times 5 + 30 \times 12.5 + 75 \times 30}{125} = \frac{2725}{125} = 21.8.$$

The median, 20, is not the mean: the long last class pulls the mean upwards.

Question 15. Answer **E**.

Only the number of times each vector is used matters, not the order. Using $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ respectively a , b , c times, we need $2a - b + c = 11$ and $a + 3b - c = 9$. Adding eliminates c : $3a + 2b = 20$, so with $a, b \geq 0$ the pair (a, b) is $(0, 10)$, $(2, 7)$, $(4, 4)$ or $(6, 1)$, and then $c = 11 - 2a + b$ is 21, 14, 7 or 0.

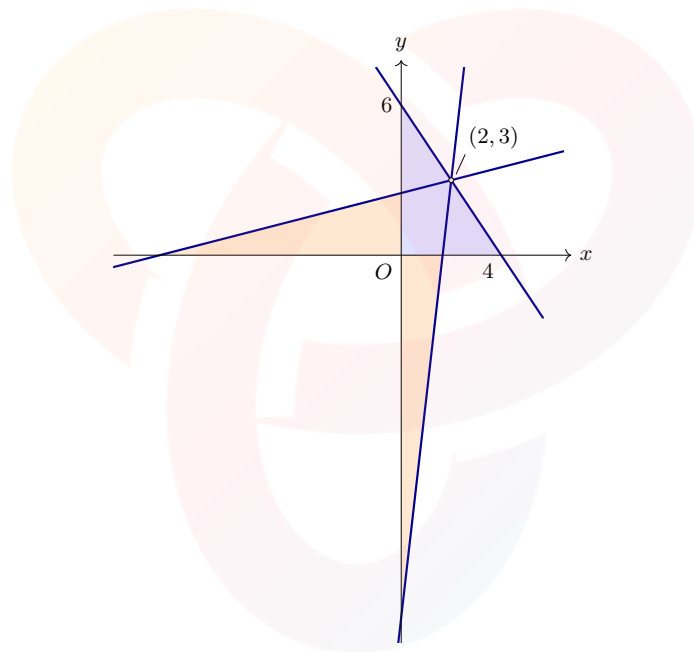
The total numbers of moves $a + b + c$ are 31, 23, 15 and 7, so the least is 7: six uses of $\mathbf{p}$ and one of $\mathbf{q}$, since $6\mathbf{p} + \mathbf{q} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$.

Question 16. Answer **E**.

Record each minute as $+1$ (clockwise) or -1 (anticlockwise). All $2^6 = 64$ sequences are equally likely. She is back at the start exactly when the total is a multiple of 6, so 0 or ± 6 .

A total of 0 needs three steps each way: $\binom{6}{3} = 20$ sequences. A total of ± 6 means walking all the way round in one direction: 2 sequences. The probability is $\frac{22}{64} = \frac{11}{32}$. Forgetting the two complete laps gives $\frac{20}{64} = \frac{5}{16}$.

Question 17. Answer **B**.



In the figure, the blue triangle comes from the double root and the two orange triangles from the other quadratic.

Vertical and horizontal lines, and the line through the origin, do not form a triangle, so take the line $y - 3 = m(x - 2)$ with $m \neq 0$. Its intercepts are $x = 2 - \frac{3}{m}$ and $y = 3 - 2m$, and their product is $\left(2 - \frac{3}{m}\right)(3 - 2m) = 12 - 4m - \frac{9}{m}$. The area is 12 when this product is 24 or -24 .

Product 24: $4m^2 + 12m + 9 = 0$, so $(2m + 3)^2 = 0$ and $m = -\frac{3}{2}$, a single line (in fact 12 is the least possible area of a triangle in the first quadrant cut off by a line through $(2, 3)$, and only this line achieves it). Product -24 : $4m^2 - 36m + 9 = 0$, whose discriminant $1296 - 144$ is positive, giving two lines with positive gradient. That is 3 lines in total. Ignoring the sign of the intercepts gives 1, and expecting two roots from each quadratic gives 4.

Question 18. Answer **D**.

The fast clock gains 5 minutes per real hour on the slow one. Two 12-hour dials show the same time when the gap between the clocks is a whole number of 12 hours, which is 720 minutes. This first

happens after $\frac{720}{5} = 144$ hours, exactly 6 days, when the real time is noon again.

By then the fast clock has gained $3 \times 144 = 432$ minutes, which is 7 hours 12 minutes, so it shows 7:12. Check: the slow clock has lost 288 minutes, which is 4 hours 48 minutes, and 12 hours minus 4 hours 48 minutes is also 7:12. The answer 12:00 is the real time, not what the clocks show.

Question 19. Answer **C**.

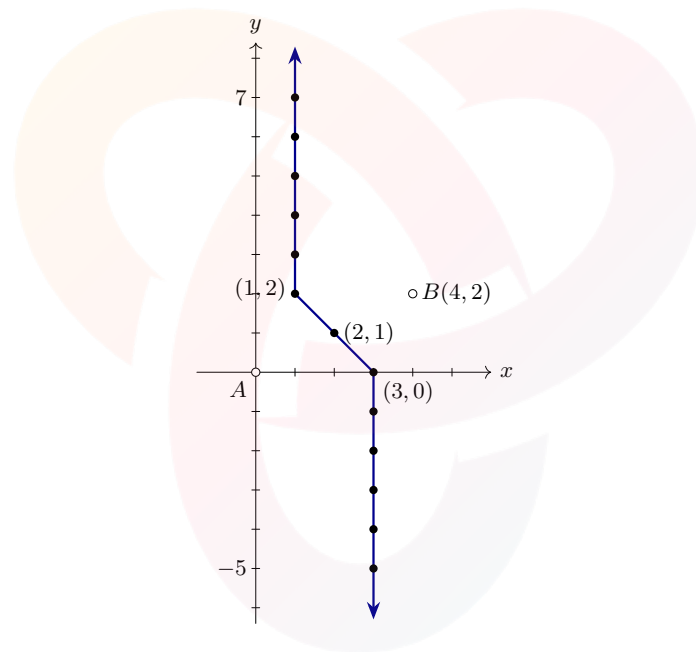
Find T as a formula. The reflection sends (x, y) to (y, x) . The rotation through 90° clockwise about $(2, 0)$ sends (u, v) to $(2 + v, 2 - u)$. So

$$T(x, y) = (2 + x, 2 - y),$$

a translation of 2 units to the right combined with a reflection in the line $y = 1$.

Applying T twice gives $(x, y) \mapsto (x + 4, y)$, a translation of 4 units to the right. So $P_{2k} = (1 + 4k, 0)$, and $P_{2026} = (1 + 4 \times 1013, 0) = (4053, 0)$. The point $(4051, 2)$ is P_{2025} , the off-by-one trap.

Question 20. Answer **C**.



Solve $|x| + |y| = |x - 4| + |y - 2|$ region by region. For $y \geq 2$ the y terms differ by exactly 2, so $|x| - |x - 4| = -2$, which gives $x = 1$: a vertical ray upwards from $(1, 2)$. For $y \leq 0$, similarly $|x| - |x - 4| = 2$, giving $x = 3$: a vertical ray downwards from $(3, 0)$. For $0 \leq y \leq 2$ the equation becomes $x + y = 3$ with $1 \leq x \leq 3$: the segment from $(1, 2)$ to $(3, 0)$.

The common distance is $1 + y$ on the upper ray, 3 on the segment and $3 + |y|$ on the lower ray. With distance at most 8: the upper ray gives $y = 2, \dots, 7$ (6 points), the segment adds $(2, 1)$, and the lower ray gives $y = 0, -1, \dots, -5$ (6 points). The total is 13. Unlike the ordinary perpendicular bisector, this set of equidistant points bends.

Question 21. Answer **E**.

Removing one ticket from $1, 2, \dots, n$ leaves a mean between $\frac{n}{2}$ (when n is removed) and $\frac{n+2}{2}$ (when 1 is removed). So $\frac{n}{2} \leq 31.25 \leq \frac{n+2}{2}$, giving $60.5 \leq n \leq 62.5$, so n is 61 or 62.

The total of the remaining tickets, $31.25(n - 1) = \frac{125}{4}(n - 1)$, must be a whole number, so $n - 1$ is a multiple of 4 and $n = 61$. The remaining total is $31.25 \times 60 = 1875$, while $1 + 2 + \dots + 61 = \frac{1}{2} \times 61 \times 62 = 1891$. The lost ticket is $1891 - 1875 = 16$.

Question 22. Answer **C**.

There is no formula to look up, so generate terms and look for a pattern. From $u_{20} = 4$ and $u_{21} = 6$:

$$u_{22} = \frac{6+1}{4} = \frac{7}{4}, \quad u_{23} = \frac{\frac{7}{4}+1}{6} = \frac{11}{24}, \quad u_{24} = \frac{\frac{11}{24}+1}{\frac{7}{4}} = \frac{5}{6}, \quad u_{25} = \frac{\frac{5}{6}+1}{\frac{11}{24}} = 4, \quad u_{26} = \frac{4+1}{\frac{5}{6}} = 6.$$

Since each term is determined by the two before it, and u_{25}, u_{26} equal u_{20}, u_{21} , the sequence repeats with period 5 from then on.

Now $2028 - 23 = 2005 = 5 \times 401$, so $u_{2028} = u_{23} = \frac{11}{24}$. Miscounting the position in the cycle by one gives $\frac{7}{4}$ or $\frac{5}{6}$, and $\frac{24}{11}$ inverts the fraction.

Question 23. Answer **B**.

Work backwards. Every number displayed is positive, a press of F always gives a number greater than 1, and a press of G always gives a number less than 1. So from any displayed number other than the starting 1, the previous number is forced: if the number exceeds 1 the last press was F (subtract 1), and if it is less than 1 the last press was G (undo it by $y \mapsto \frac{y}{1-y}$).

This gives

$$\frac{29}{12} \leftarrow \frac{17}{12} \leftarrow \frac{5}{12} \leftarrow \frac{5}{7} \leftarrow \frac{5}{2} \leftarrow \frac{3}{2} \leftarrow \frac{1}{2} \leftarrow 1,$$

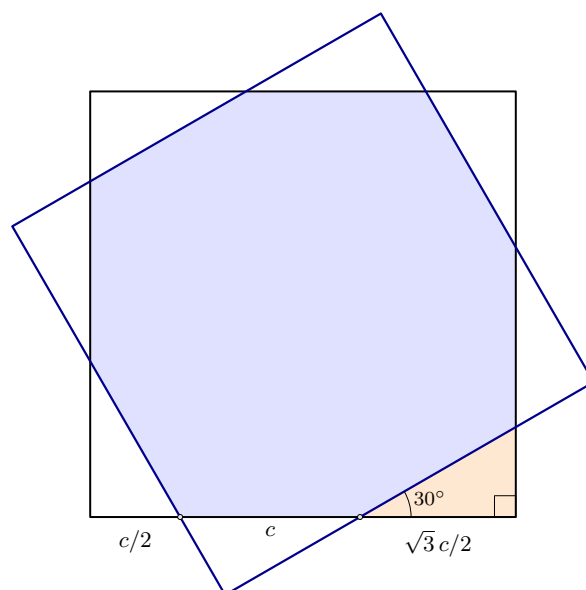
which is 7 presses (in order G, F, F, G, G, F, F). Since every backward step was forced, there is no other route, let alone a shorter one. The process is the Euclidean algorithm in disguise: $29 = 2 \times 12 + 5$, $12 = 2 \times 5 + 2$, $5 = 2 \times 2 + 1$, $2 = 2 \times 1$.

Question 24. Answer **E**.

Look at the first two tosses. If they are both heads, Ana wins at once, with probability $\frac{1}{4}$. Otherwise a tail has appeared by the second toss, and from then on Ana cannot win: she needs a head that comes straight after another head, but the first head after any tail comes straight after a tail, which completes Ben's pattern and ends the game.

So Ana wins exactly when the first two tosses are heads, and the probability is $\frac{1}{4}$. The tempting $\frac{1}{2}$ assumes that the two patterns, each of probability $\frac{1}{4}$ at any given pair of tosses, are equally strong.

Question 25. Answer **A**.



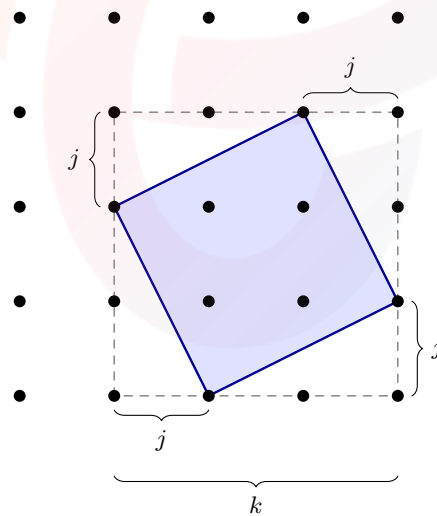
The common region is an octagon, obtained from either square by cutting off a small right-angled triangle at each of its four corners. Each cut is made by a side of the other square, which is at 30° to the sides it crosses, so each corner triangle has angles 30° , 60° , 90° . By symmetry all eight such triangles (four from each square) are congruent; call their hypotenuse c , so their legs are $\frac{1}{2}c$ and $\frac{\sqrt{3}}{2}c$. Each side of the first square consists of a leg of one corner triangle, a side of the octagon (which is the hypotenuse of a corner triangle of the second square) and a leg of another corner triangle. So $\frac{1}{2}c + c + \frac{\sqrt{3}}{2}c = 1$, giving $c = \frac{2}{3 + \sqrt{3}} = \frac{3 - \sqrt{3}}{3}$ and $c^2 = \frac{4 - 2\sqrt{3}}{3}$. The four corner triangles have total area $4 \times \frac{1}{2} \times \frac{1}{2}c \times \frac{\sqrt{3}}{2}c = \frac{\sqrt{3}}{2}c^2 = \frac{2\sqrt{3} - 3}{3}$, so the common area is $1 - \frac{2\sqrt{3} - 3}{3} = 2 - \frac{2\sqrt{3}}{3} \approx 0.845$. The value $2\sqrt{2} - 2$ is the answer for a 45° rotation.

Question 26. Answer **A**.

In a connected row, the numbers written so far always form a block of consecutive integers: a new number differs by 1 from something already written and is not already in the block, so it must be one less than the smallest or one more than the largest. Conversely, any row built by extending the block at one end or the other is connected.

Read the row from the right. The last number must be an end of the full block, so it is 1 or 7: two choices. Removing it leaves a block, and the second last number must be one of its two ends, and so on. There are two choices for each of the last six numbers, and the first number is then forced, giving $2^6 = 64$ rows. (Equivalently, starting with a there are $\binom{6}{a-1}$ ways to interleave the downward and upward extensions, and these sum to 2^6 .)

Question 27. Answer **F**.



Every such square has an upright bounding square (sides parallel to the rows and columns) whose corners are dots, and each vertex of the square lies on a different side of the bounding square. If the bounding square has side k cm, then going round it, the vertices sit at the same distance j along each side, where $j = 0, 1, \dots, k - 1$ (with $j = 0$ giving the upright square itself). So each upright $k \times k$ bounding square holds exactly k squares.

There are $(5 - k)^2$ upright squares of side k in the array, so the total is

$$\sum_{k=1}^4 k(5 - k)^2 = 1 \times 16 + 2 \times 9 + 3 \times 4 + 4 \times 1 = 50.$$

Counting only the upright squares gives $16 + 9 + 4 + 1 = 30$.