

HU Collection ESAT Mock Set C: Mathematics 2

Worked Solutions

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Answer key

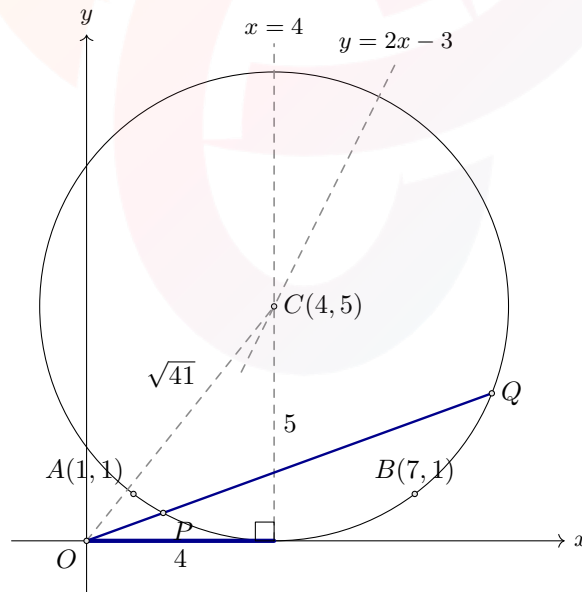
Q	1	2	3	4	5	6	7	8	9	10
Ans	E	F	B	F	F	C	A	C	D	B

Q	11	12	13	14	15	16	17	18	19	20
Ans	D	B	E	E	C	B	A	F	A	B

Q	21	22	23	24	25	26	27			
Ans	C	E	A	D	C	A	D			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer **E**.



The centre lies on the perpendicular bisector of AB , which is $x = 4$, and on $y = 2x - 3$, so it is $C(4, 5)$, with $r^2 = CA^2 = 9 + 16 = 25$. For any line through O meeting the circle at P and Q , the product $OP \times OQ$ is the same (intersecting secants), and taking the tangent from O shows it equals the square of the tangent length, $OC^2 - r^2 = 41 - 25 = 16$. The value 41 is OC^2 (forgetting to subtract r^2), and 25 is r^2 ; the product does not depend on the direction.

Question 2. Answer **F**.

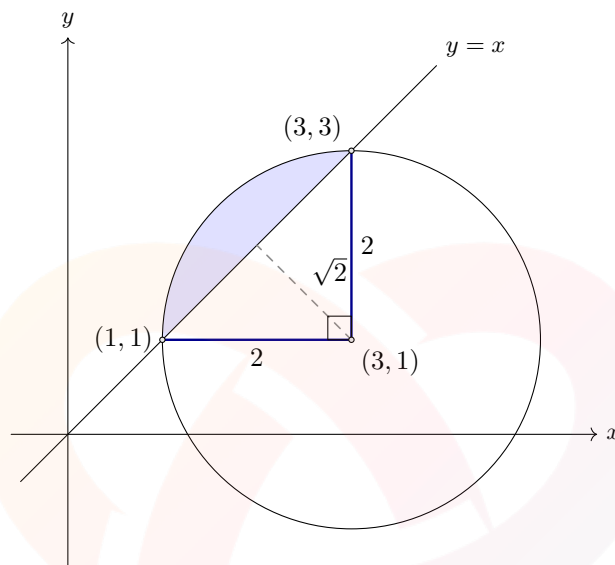
Successive differences of $p + qr^t$ are $qr^t(r - 1)$, so they are in geometric progression with ratio r . Here the differences are -6 and -3 , so $r = \frac{1}{2}$. Then $qr(r - 1) = -6$ gives $q \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) = -6$, so

$q = 24$, and $p = 10 - 24 \cdot \frac{1}{2} = -2$. Since $0 < r < 1$, $qr^t \rightarrow 0$ and the readings tend to $p = -2$. The tempting answer 0 assumes the readings themselves decay to zero; $-\frac{1}{2}$ is the next reading $f(4)$.

Question 3. Answer **B**.

The general term of $(2x^2 - \frac{1}{x})^9$ is $\binom{9}{k}(2x^2)^{9-k}(-\frac{1}{x})^k = \binom{9}{k}2^{9-k}(-1)^k x^{18-3k}$. The constant term of the product is the constant term of this expansion plus its coefficient of x^{-3} . For x^0 , $k = 6$: $\binom{9}{6}2^3 = 84 \times 8 = 672$. For x^{-3} , $k = 7$: $\binom{9}{7}2^2(-1)^7 = -144$. The total is $672 - 144 = 528$. Forgetting the x^3 partner gives 672; losing the sign of $(-1)^7$ gives 816.

Question 4. Answer **F**.



Completing the square, $(x - 3)^2 + (y - 1)^2 = 4$: centre $(3, 1)$, radius 2. The distance from the centre to $x - y = 0$ is $\frac{|3-1|}{\sqrt{2}} = \sqrt{2}$, so the half-chord is $\sqrt{4-2} = \sqrt{2}$ and the chord subtends a right angle at the centre. The minor segment has area $\frac{1}{2} \cdot 2^2 \cdot \frac{\pi}{2} - \frac{1}{2} \cdot 2^2 \sin \frac{\pi}{2} = \pi - 2$. The value $3\pi + 2$ is the major segment.

Question 5. Answer **F**.

Write $a = \log_2 x$ and $k = \log_2 3$, so $\log_3 x = \frac{a}{k}$. The equation becomes $\frac{a^2}{k} = a + \frac{a}{k}$, that is $a(\frac{a}{k} - 1 - \frac{1}{k}) = 0$. So $a = 0$, giving $x = 1$, or $a = k + 1 = \log_2 3 + \log_2 2 = \log_2 6$, giving $x = 6$. The sum is 7. Dividing through by $\log_2 x$ at the start loses $x = 1$ and leads to 6.

Question 6. Answer **C**.

The roots come in pairs $\pm\alpha$, $\pm\beta$ with $0 < \alpha < \beta$, and in increasing order they are $-\beta, -\alpha, \alpha, \beta$. The common difference is 2α , so $\beta = 3\alpha$. Then α^2 and $9\alpha^2$ are the roots of $u^2 - 10u + k = 0$, so $10\alpha^2 = 10$, $\alpha^2 = 1$ and $k = \alpha^2 \cdot 9\alpha^2 = 9$. Assuming the roots are $\pm a, \pm 2a$ (gaps not equal) gives 16.

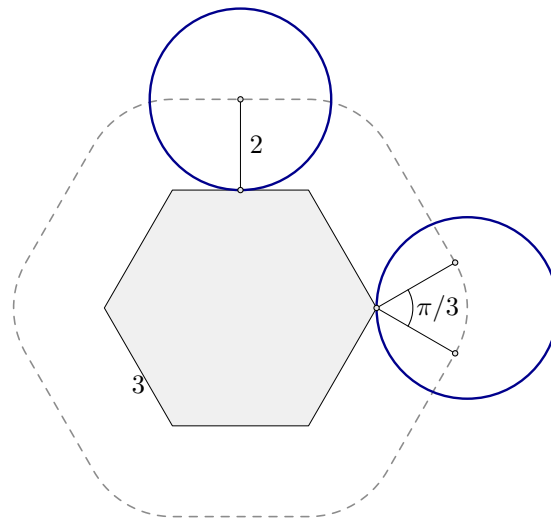
Question 7. Answer **A**.

With first term a and ratio r , $S_4 = S_\infty(1 - r^4)$, so $16(1 - r^4) = 15$ and $r^4 = \frac{1}{16}$, giving $r = \pm\frac{1}{2}$. If $r = \frac{1}{2}$ then $a = 8$; if $r = -\frac{1}{2}$ then $a = 24$, which is also positive. The odd-position terms form a geometric series with ratio $r^2 = \frac{1}{4}$, so $T = \frac{a}{1-r^2} = \frac{4}{3}a$, which is $\frac{32}{3}$ or 32. The sum is $\frac{128}{3}$. Missing the negative ratio gives $\frac{32}{3}$.

Question 8. Answer **C**.

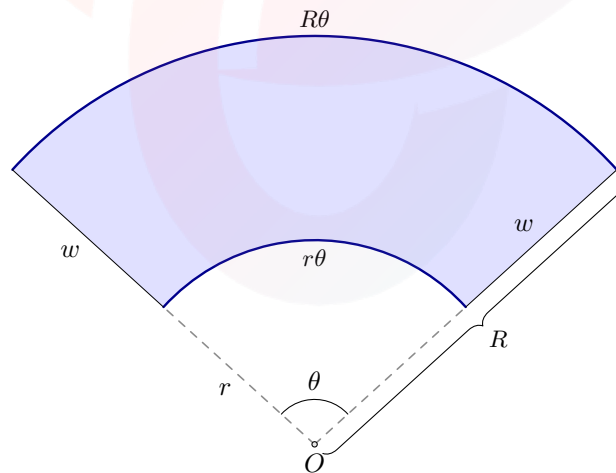
Note $\sqrt{9x+18} = 3\sqrt{x+2} = \sqrt{9(x+2)}$. I: $\sqrt{x+2}$, then $3\sqrt{x+2}$; correct. II: $\sqrt{9x}$, then $\sqrt{9(x+2)}$; correct. III: $\sqrt{x+18}$, then replacing x by $9x$ gives $\sqrt{9x+18}$; correct. IV: $\sqrt{x+2}$, then $\sqrt{9x+2}$; wrong, because a stretch applied after the translation also scales the shift. So I, II and III only.

Question 9. Answer **D**.



Along each side the wheel rolls a distance 3, turning through $\frac{3}{2}$ radians, giving 9 for the six sides. At each vertex the point of contact stays at the vertex while the wheel swings round it through the exterior angle $\frac{\pi}{3}$, and the wheel turns through that same angle; six vertices add 2π . The total is $9 + 2\pi$. As a check, the centre travels $18 + 2\pi \cdot 2$, and dividing by the radius 2 gives $9 + 2\pi$. Ignoring the corners gives 9.

Question 10. Answer **B**.



Let $L = \theta(R + r)$ be the total length of the two arcs and $w = R - r$ the length of each straight edge. The area is $\frac{1}{2}\theta(R^2 - r^2) = \frac{1}{2}\theta(R + r)(R - r) = \frac{1}{2}Lw$, like a trapezium. The budget gives $2L + 10w = 200$, so $L = 100 - 5w$ and the area is $\frac{1}{2}w(100 - 5w)$, which is greatest at $w = 10$, $L = 50$, giving 250. This is achievable, for example with $r = 10$, $R = 20$, $\theta = \frac{5}{3}$. Forgetting the factor $\frac{1}{2}$ gives 500.

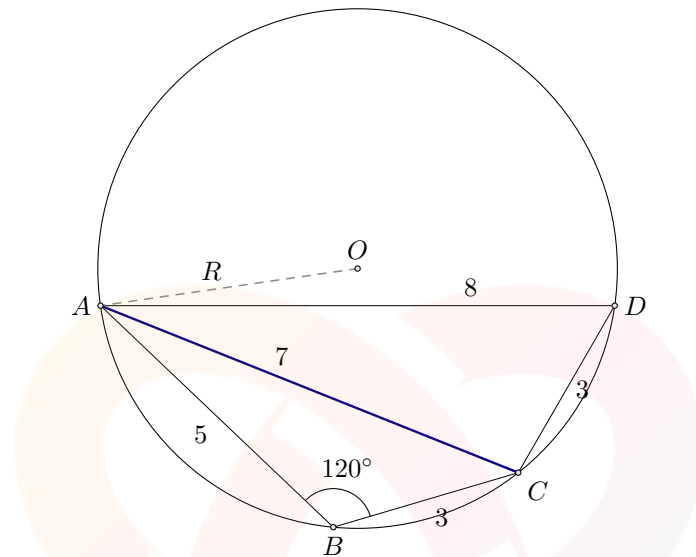
Question 11. Answer **D**.

Put $u = 2x + \frac{\pi}{6}$, which runs over $\frac{\pi}{6} \leq u \leq \frac{13\pi}{6}$, exactly one period. On this interval $\sin u$ starts at $\frac{1}{2}$, rises to 1, falls to -1 and returns to $\frac{1}{2}$. A horizontal line at height k meets this once for $k = \pm 1$, twice for $-1 < k < 1$ with $k \neq \frac{1}{2}$, and three times for $k = \frac{1}{2}$ (at both ends and at $u = \frac{5\pi}{6}$). The value $k = \frac{1}{2}$ is the trap.

Question 12. Answer **B**.

Any circle containing A and B has a diameter at least $AB = 8$, so the radius is at least 4. The circle with AB as diameter has centre $(4, 0)$ and radius 4, and C is at distance $\sqrt{1+4} = \sqrt{5} < 4$ from its centre, so it encloses all three trees. The answer is 4. The trap is the circumcircle: the angle ACB is obtuse (since $\vec{CA} \cdot \vec{CB} = -15 + 4 < 0$), and for an obtuse triangle the circumcircle, of radius $\frac{abc}{4\Delta} = \frac{8\sqrt{13}\sqrt{29}}{32} = \frac{\sqrt{377}}{4}$, is larger than necessary.

Question 13. Answer **E**.



Opposite angles of a cyclic quadrilateral are supplementary, so $\cos D = -\cos B$. The cosine rule for the diagonal AC in triangles ABC and ADC gives $25 + 9 - 30 \cos B = 9 + 64 + 48 \cos B$, so $\cos B = -\frac{1}{2}$, $B = 120^\circ$ and $AC^2 = 34 + 15 = 49$, $AC = 7$. The circumcircle of $ABCD$ is that of triangle ABC , so by the sine rule $2R = \frac{7}{\sin 120^\circ}$ and $R = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$. A sign slip giving $\cos B = \frac{1}{2}$ leads to $AC = \sqrt{19}$ and $\frac{\sqrt{57}}{3}$; treating AC as a diameter gives $\frac{7}{2}$.

Question 14. Answer **E**.

Expanding, $x^n(1-x)^2 = x^n - 2x^{n+1} + x^{n+2}$, so $A_n = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3} = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$. Summing from $n = 1$, both brackets telescope: the first gives $\frac{1}{2}$ and the second $\frac{1}{3}$, so the total is $\frac{1}{6}$. Starting the sum at $n = 0$ by mistake adds $A_0 = \frac{1}{3}$ and gives $\frac{1}{2}$.

Question 15. Answer **C**.

The coefficients are non-negative integers with sum $p(1) = 7$, so each coefficient is at most 7, less than 10. Hence $p(10)$ written in base ten has the coefficients as its digits: 2041 means $p(x) = 2x^3 + 0x^2 + 4x + 1$. Then $p(3) = 54 + 12 + 1 = 67$. Reading the digits as $2x^3 + 4x^2 + 1$ gives 91.

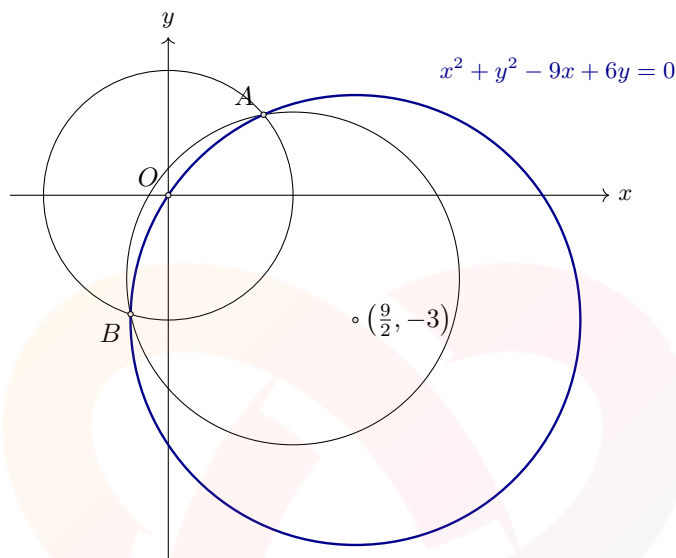
Question 16. Answer **B**.

Reading each row from left to right as the coefficients of ascending powers of x , the rule says row $n + 1$ is row n multiplied by $(2 + x)$: the left neighbour contributes x times, the right one 2 times. So row 9 lists the coefficients $\binom{8}{k}2^{8-k}$ of $(2 + x)^8$: 256, 1024, 1792, 1792, 1120, 448, 112, 16, 1. The largest is 1792 (it occurs twice). The value $1120 = \binom{8}{4}2^4$ comes from assuming the middle entry is largest; 5376 is the largest entry of row 10.

Question 17. Answer **A**.

Reflect the pond in the y -axis and then in the x -axis: its centre moves to $C''(-2, -3)$. A walk $A \rightarrow P \rightarrow Q \rightarrow \text{pond}$, with P on the x -axis and Q on the y -axis, has the same length as the broken line from A via P and the reflected copy of Q to the doubly reflected pond, which is at least $AC'' - 1 = \sqrt{9 + 16} - 1 = 4$. The straight line from A to C'' crosses the x -axis at $(\frac{1}{4}, 0)$ and then the reflected y -axis, giving the real point $(0, \frac{1}{3})$, so 4 is achieved: $\frac{5}{4} + \frac{5}{12} + (\frac{10}{3} - 1) = 4$. Visiting the y -axis first unfolds to the same point C'' , so it cannot do better. Going via the origin gives $\sqrt{2} + \sqrt{13} - 1 \approx 4.02$, slightly longer.

Question 18. Answer **F**.

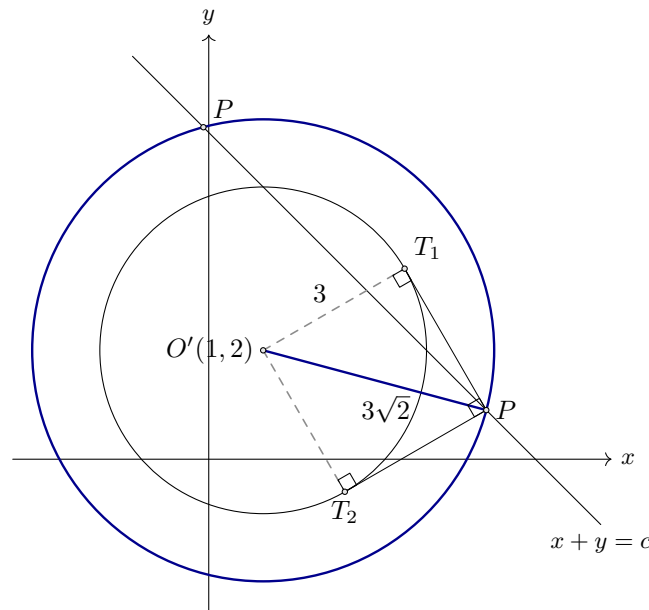


Finding A and B explicitly is messy. Instead, every curve $(x^2 + y^2 - 9) + \lambda(x^2 + y^2 - 6x + 4y - 3) = 0$ passes through A and B , and for $\lambda \neq -1$ it is a circle. Through the origin: $-9 - 3\lambda = 0$, so $\lambda = -3$, giving $-2x^2 - 2y^2 + 18x - 12y = 0$, that is $x^2 + y^2 - 9x + 6y = 0$. Its centre is $(\frac{9}{2}, -3)$ and $r^2 = \frac{81}{4} + 9 = \frac{117}{4}$, so $r = \frac{3\sqrt{13}}{2}$. (The circles do meet: their centres are $\sqrt{13}$ apart and $1 < \sqrt{13} < 7$.)

Question 19. Answer **A**.

By the cosine rule, $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{c^2}{2ab} \geq \frac{c^2}{a^2 + b^2} = \frac{1}{2}$, using $2ab \leq a^2 + b^2$, so $C \leq 60^\circ$, with equality for the equilateral triangle. Conversely, scale so that $c = 1$; then $a^2 + b^2 = 2$ and $\cos C = \frac{1}{2ab}$. The triangle inequality $|a - b| < 1$ is equivalent to $2 - 2ab < 1$, that is $ab > \frac{1}{2}$, and ab can take every value in $(\frac{1}{2}, 1]$, so $\cos C$ takes every value in $[\frac{1}{2}, 1)$. Hence $0^\circ < C \leq 60^\circ$. Spotting only the equilateral case gives $C = 60^\circ$ only.

Question 20. Answer **B**.

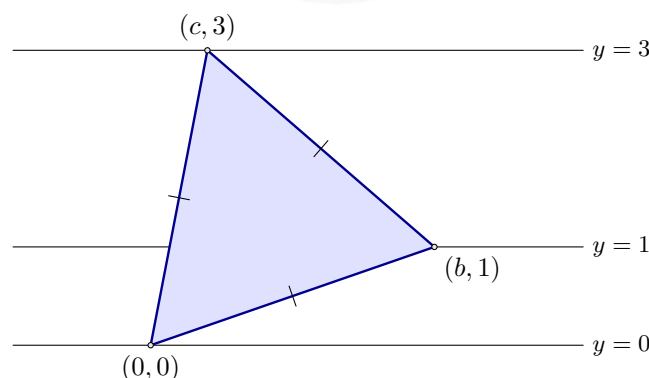


C has centre $O'(1, 2)$ and radius 3. If the tangents from P touch C at T_1 and T_2 and are perpendicular, then $O'T_1PT_2$ has four right angles and two adjacent sides equal to 3, so it is a square and $O'P = 3\sqrt{2}$. Conversely every point at distance $3\sqrt{2}$ from O' works. So the points P lie on the circle with centre $(1, 2)$ and radius $3\sqrt{2}$, and there are exactly two of them when the line is closer to $(1, 2)$ than $3\sqrt{2}$: $\frac{|1+2-c|}{\sqrt{2}} < 3\sqrt{2}$, that is $|c - 3| < 6$, so $c = -2, -1, \dots, 8$, giving 11 values. Using radius 3 instead of $3\sqrt{2}$ in the distance condition gives 9.

Question 21. Answer **C**.

Let $u = \sin^2 x$, so $\cos^2 x = 1 - u$ with $0 \leq u \leq 1$. Since $s^3 + c^3 = (s + c)^3 - 3sc(s + c)$ with $s + c = 1$, the left side is $1 - 3u(1 - u)$. The equation becomes $3u - 3u^2 = \frac{3}{8}$, that is $8u^2 - 8u + 1 = 0$, so $u = \frac{2 \pm \sqrt{2}}{4}$. Both values lie between 0 and 1 (and neither is 0 or 1), and each gives $\sin x = \pm\sqrt{u}$, which is four distinct values of x in $[0, 2\pi)$. So there are 8 solutions. Stopping at the two values of u , or taking only positive square roots, gives 2 or 4.

Question 22. Answer **E**.



Place the vertices at $(0, 0)$, $(b, 1)$ and $(c, 3)$. Equal sides give $b^2 + 1 = c^2 + 9$ and $b^2 + 1 = (c - b)^2 + 4$. The second gives $2bc = c^2 + 3$, so $b = \frac{c^2 + 3}{2c}$, and substituting in $b^2 = c^2 + 8$ gives $(c^2 + 3)^2 = 4c^2(c^2 + 8)$, that is $3c^4 + 26c^2 - 9 = 0$, or $(3c^2 - 1)(c^2 + 9) = 0$. So $c^2 = \frac{1}{3}$, the square of the side is $c^2 + 9 = \frac{28}{3}$ whatever the configuration, and the area is $\frac{\sqrt{3}}{4} \cdot \frac{28}{3} = \frac{7\sqrt{3}}{3}$. Taking the side to be 3 (as if one side were vertical) gives $\frac{9\sqrt{3}}{4}$.

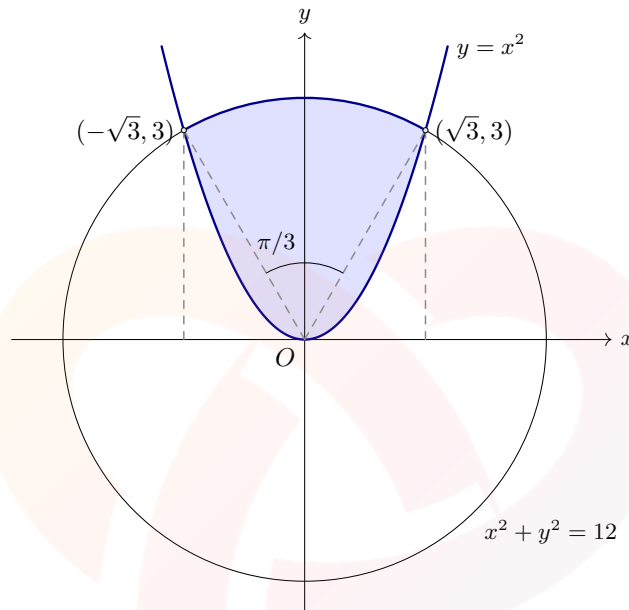
Question 23. Answer **A**.

Adding 51 separate coefficients $\binom{50}{50} + \binom{51}{50} + \dots + \binom{100}{50}$ directly is hopeless. Instead treat the sum as a geometric series with ratio $1+x$: it equals

$$\frac{(1+x)((1+x)^{100}-1)}{(1+x)-1} = \frac{(1+x)^{101}-(1+x)}{x}.$$

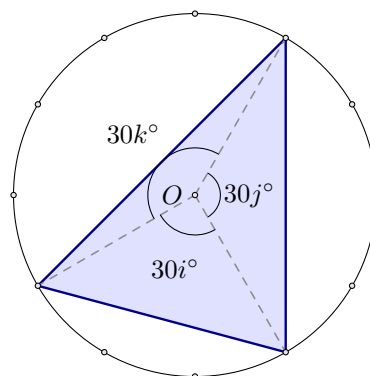
The coefficient of x^{50} is the coefficient of x^{51} in $(1+x)^{101} - (1+x)$, which is $\binom{101}{51}$. Taking only the last term gives $\binom{100}{50}$.

Question 24. Answer **D**.



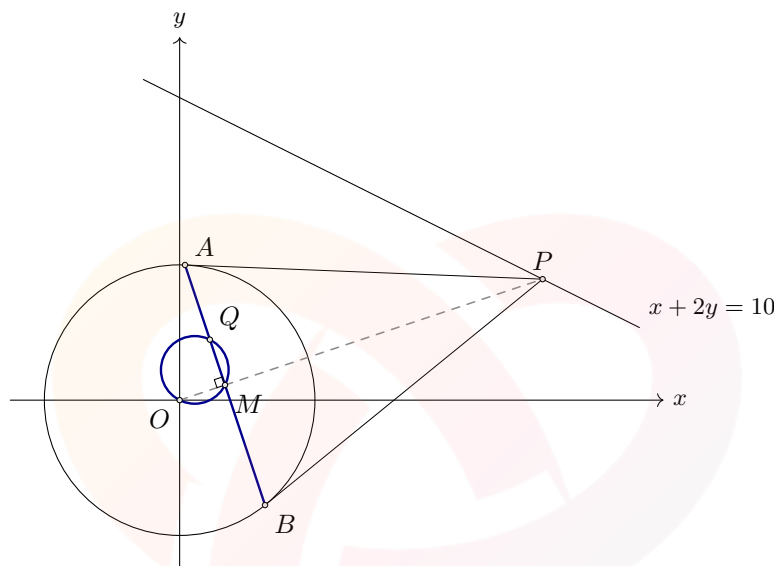
The curves meet where $y + y^2 = 12$, so $y = 3$ and $x = \pm\sqrt{3}$. The region above the parabola and inside the circle has area $\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{12-x^2} dx - \int_{-\sqrt{3}}^{\sqrt{3}} x^2 dx$. The first integral cannot be found by our methods, so use geometry: the points $(\pm\sqrt{3}, 3)$ are at angles 60° and 120° from the origin, so the area under the arc above $-\sqrt{3} \leq x \leq \sqrt{3}$ is a sector of radius $\sqrt{12}$ and angle $\frac{\pi}{3}$, area 2π , plus two right-angled triangles with legs $\sqrt{3}$ and 3, area $3\sqrt{3}$ in total. The second integral is $\frac{2(\sqrt{3})^3}{3} = 2\sqrt{3}$. So this region has area $2\pi + \sqrt{3} \approx 8.0$, while the disc has area $12\pi \approx 37.7$, so it is the smaller region. Forgetting to subtract the area under the parabola gives $2\pi + 3\sqrt{3}$, taking the sector angle as $\frac{2\pi}{3}$ gives $4\pi + \sqrt{3}$, and $10\pi - \sqrt{3}$ is the larger region.

Question 25. Answer **C**.



The three vertices split the circle into arcs subtending $30i^\circ$, $30j^\circ$, $30k^\circ$ at the centre O , with $i + j + k = 12$. The triangle is made of the three triangles with apex O (with a sign change for an obtuse triangle), and in every case its area is $\frac{1}{2}(\sin 30i^\circ + \sin 30j^\circ + \sin 30k^\circ)$. The values of $\sin 30m^\circ$ are $\pm\frac{1}{2}$, ± 1 or 0 , except that $m = 2, 4$ give $\frac{\sqrt{3}}{2}$ and $m = 8, 10$ give $-\frac{\sqrt{3}}{2}$. So the area is rational exactly when the parts equal to 2 or 4 are as many as the parts equal to 8 or 10. Of the twelve partitions of 12 into three positive parts, only $\{1, 5, 6\}$ and $\{3, 3, 6\}$ qualify. There are 24 triangles of the first type and 12 of the second, so 36 in all. Note that the triangles of type $\{2, 4, 6\}$ are right-angled but have area $\frac{\sqrt{3}}{2}$, so “all 60 right-angled triangles” is wrong.

Question 26. Answer A.



For $P(p, q)$, the points A and B lie on the circle with diameter OP (the angles OAP and OBP are right angles), which is $x^2 + y^2 - px - qy = 0$. Subtracting from $x^2 + y^2 = 5$ shows that the line AB is $px + qy = 5$. With $p = 10 - 2q$ this reads $(10x - 5) + q(y - 2x) = 0$, so every chord AB passes through the fixed point $Q(\frac{1}{2}, 1)$. Now M is the foot of the perpendicular from O to AB , so $\angle OMQ = 90^\circ$ and M lies on the circle with diameter $OQ = \frac{\sqrt{5}}{2}$; every point of that circle except O itself occurs. The length is $\pi \cdot \frac{\sqrt{5}}{2}$. Using the distance $2\sqrt{5}$ from O to the line as a diameter gives $2\sqrt{5}\pi$.

Question 27. Answer D.

Write the left side as $\sin(\frac{\pi}{2} - \pi \sin x)$. Then $\sin A = \sin B$ means $A = B + 2k\pi$ or $A = \pi - B + 2k\pi$ for an integer k . With $A = \pi \cos x$ and $B = \frac{\pi}{2} - \pi \sin x$ this gives $\cos x + \sin x = \frac{1}{2} + 2k$ or $\cos x - \sin x = \frac{1}{2} + 2k$. Writing $u = \cos x$, $v = \sin x$, the point (u, v) is on the unit circle, where $u \pm v$ takes only values between $-\sqrt{2}$ and $\sqrt{2}$; so only $\frac{1}{2}$ is possible ($-\frac{3}{2} < -\sqrt{2}$ and $\frac{5}{2} > \sqrt{2}$). Each of the lines $u + v = \frac{1}{2}$ and $u - v = \frac{1}{2}$ is at distance $\frac{1}{2\sqrt{2}} < 1$ from the origin, so meets the unit circle twice, and the two lines share only the point $(\frac{1}{2}, 0)$, which is not on the circle. Each point of the circle corresponds to one x in $[0, 2\pi)$, so there are 4 solutions. Including $u + v = -\frac{3}{2}$ as if it were attainable gives 6 or 8.