

HU Collection

ESAT Mock Set C: Mathematics 1

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	B	F	A	C	C	E	C	D	E	A

Q	11	12	13	14	15	16	17	18	19	20
Ans	F	C	B	D	E	F	D	B	A	B

Q	21	22	23	24	25	26	27			
Ans	E	C	A	F	A	B	D			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer B.

First, $5.4 \text{ km/h} = 5400 \text{ m} \div 3600 \text{ s} = 1.5 \text{ m/s}$. The hose delivers $\pi(0.01)^2 \times 1.5 = 1.5 \times 10^{-4} \pi \text{ m}^3$ per second, and the pool needs $\pi(1)^2 \times 0.3 = 0.3\pi \text{ m}^3$. The factors of π cancel, and the time is $0.3 \div (1.5 \times 10^{-4}) = 2000$ seconds, which is 33 minutes 20 seconds. Using the hose diameter as its radius gives 8 minutes 20 seconds; using the pool diameter as its radius gives 2 hours 13 minutes 20 seconds; a slip of a factor of 10 in the speed conversion gives 3 minutes 20 seconds or 5 hours 33 minutes 20 seconds.

Question 2. Answer F.

Over a distance of d miles a van doing m mpg uses $\frac{d}{m}$ gallons, so the saving is $d \left(\frac{1}{m_1} - \frac{1}{m_2} \right)$.

This gives $\frac{d}{84}$ for R , $\frac{d}{96}$ for P , $\frac{d}{120}$ for Q and $\frac{d}{180}$ for S , so the order is R, P, Q, S . The trap Q, P, S, R orders the vans by the gain in mpg (which here is also the order of percentage gain). Fuel used is the reciprocal of the mpg figure, so a small gain at a low mpg saves far more fuel than a large gain at a high one.

Question 3. Answer A.

With edges a, b, c we have $4(a+b+c) = 76$, so $a+b+c = 19$, and by Pythagoras in three dimensions $a^2 + b^2 + c^2 = 13^2 = 169$. The surface area is

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 361 - 169 = 192 \text{ dm}^2.$$

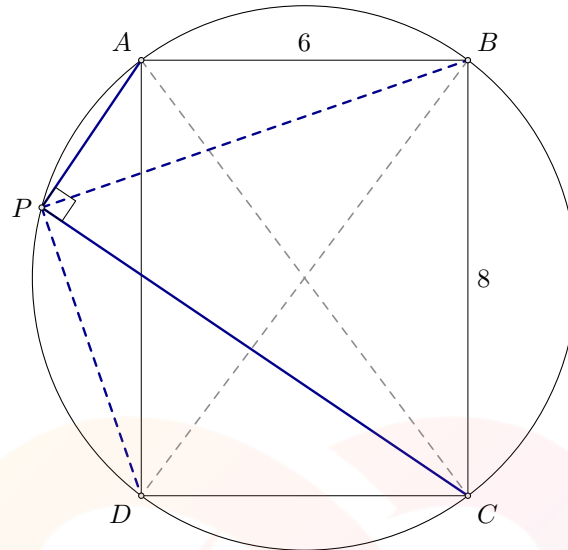
There is no need to find the edges (they are 3, 4 and 12 dm). The option 96 is $ab + bc + ca$, forgetting that each face appears twice, and 384 doubles once too often.

Question 4. Answer C.

Pressure is weight divided by area. For a uniform cuboid the weight is (weight per unit volume) $\times$ (area of the face it rests on) $\times$ (height), so the area cancels and the pressure is proportional to the height alone. The first block stands 0.5 m tall, giving 12 kPa, which is 24 kPa per metre of height. The second block measures 3 m by 1.5 m by 0.75 m and on its smallest face it stands 3 m tall, so

the pressure is 72 kPa. Equivalently, the weight is multiplied by $1.5^3 = 3.375$ and the area changes from 2 m^2 to 1.125 m^2 . The option 18 kPa applies the enlargement but not the change of face, and 48 kPa changes the face but forgets the enlargement.

Question 5. Answer C.



Since $\angle ABC = 90^\circ$, the diagonal AC is a diameter of the circle (converse of the angle in a semicircle), and so is BD ; each has length $\sqrt{6^2 + 8^2} = 10$. Now P is on the circle, so $\angle APC = 90^\circ$ and $PA^2 + PC^2 = AC^2 = 100$; likewise $PB^2 + PD^2 = BD^2 = 100$. The total is 200, wherever P is. The option 100 uses only one of the two diameters.

Question 6. Answer E.

The first differences are 6, 8, 13, 17, 21, 25 and the second differences are 2, 5, 4, 4, 4. A single wrong term disturbs only the second differences next to it, so the true second difference is 4, and the terms 17, 30, 47, 68, 93 fit $u_n = 2n^2 - n + 2$. This also gives $u_1 = 3$ but $u_2 = 8$, so the 9 is the error (no single change elsewhere makes all seven values quadratic). Then $u_{10} = 200 - 10 + 2 = 192$. The trap 129 comes from trusting the first three values: 3, 9, 17 lead to $n^2 + 3n - 1$.

Question 7. Answer C.

Group A has total 780 and group B has total 900. If the student scored s , then $\frac{780 - s}{11} \geq 65.4$ gives $s \leq 60.6$, and $\frac{900 + s}{19} \geq 50.4$ gives $s \geq 57.6$. So $s = 58, 59$ or 60 : three scores. The tempting answer 0 argues that the overall mean of all 30 students cannot change, so both means cannot rise; but moving a score that is below A's mean and above B's mean raises both. The option 14 counts every score from 51 to 64, which only makes both means rise by some amount.

Question 8. Answer D.

Outdoors Xavi has 60% against Yara's 50% regardless of n . Indoors we need $\frac{n}{30} < \frac{9}{10}$, so $n \leq 26$. Overall Xavi has $\frac{21}{30} = 70\%$, and Yara needs $\frac{5 + n}{40} > \frac{7}{10}$, so $n > 23$. Hence $n = 24, 25$ or 26 . This is Simpson's paradox: Yara shoots most of her arrows indoors, where hits are easier, and Xavi most of his outdoors. The option 0 is the intuition that winning both sessions must mean winning overall.

Question 9. Answer E.

The constraints $2x + 3y \leq 24$, $y \geq 2$ and $x \geq y$ describe the triangle with vertices $(2, 2)$, $(9, 2)$ and $(4.8, 4.8)$. The profit $4x + 7y$ would be largest at $(4.8, 4.8)$, giving 52.80, but that is not a

whole-number plan, and rounding to $(5, 5)$ needs 25 hours. Checking the best whole-number point in each row: $y = 4$ allows $x \leq 6$, giving $24 + 28 = 52$; $y = 3$ allows $x \leq 7$, giving 49; $y = 2$ allows $x \leq 9$, giving 50; and $y \geq 5$ is impossible. So the greatest profit is 52 pounds, at $(6, 4)$. The option 44 comes from rounding the vertex down to $(4, 4)$.

Question 10. Answer **A**.

Completing the square, $f(t) = (t - 3)^2 + 2$. So $f(f(x)) < 27$ means $(f(x) - 3)^2 < 25$, that is $-2 < f(x) < 8$. The left-hand inequality always holds because $f(x) \geq 2$, so we need $(x - 3)^2 + 2 < 8$, that is $(x - 3)^2 < 6$. This allows $x - 3 \in \{-2, -1, 0, 1, 2\}$, giving the five integers 1, 2, 3, 4, 5. The option 3 comes from solving $(x - 3)^2 < 4$, an error in the inner step.

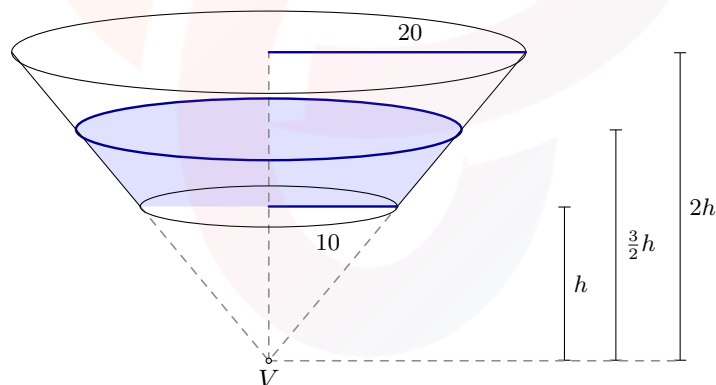
Question 11. Answer **F**.

Ali gets k when $(k - \frac{1}{2})^2 \leq n < (k + \frac{1}{2})^2$, that is (for integers) $k^2 - k + 1 \leq n \leq k^2 + k$. Bea gets k when $10k - 5 \leq n \leq 10k + 4$. For $k \leq 8$ Ali's range lies wholly below Bea's ($72 < 75$ when $k = 8$), and for $k \geq 12$ wholly above ($133 > 124$ when $k = 12$). For $k = 9$ the ranges 73 to 90 and 85 to 94 share 6 values; for $k = 10$, 91 to 110 and 95 to 104 share 10; for $k = 11$, 111 to 132 and 105 to 114 share 4. The total is 20. The trap 10 only looks near $n = 100$.

Question 12. Answer **C**.

The mean of the first two numbers is 7, so after n numbers the mean is $n + 5$ and the total is $S_n = n(n + 5)$. The 20th number is $S_{20} - S_{19} = 500 - 456 = 44$. In general the n th number is $S_n - S_{n-1} = 2n + 4$ for $n \geq 2$; for example the third is 10, and 4, 10, 10 has mean 8. The option 25 is the mean after 20 numbers, not the 20th number.

Question 13. Answer **B**.



Extend the bucket downwards to a complete cone. The radius is proportional to the distance from the vertex, so if the base is at distance h from the vertex then the rim is at $2h$ and the water surface, half way up, is at $\frac{3}{2}h$. Volumes of similar cones scale with the cube of the distance, so the bucket is proportional to $2^3 - 1 = 7$ and the water to $(\frac{3}{2})^3 - 1 = \frac{19}{8}$. The fraction is $\frac{19}{8} \div 7 = \frac{19}{56}$. The option $\frac{37}{56}$ is the answer for the wide end at the bottom, $\frac{5}{12}$ uses areas instead of volumes, and $\frac{27}{64}$ treats the water as a cone similar to the whole bucket.

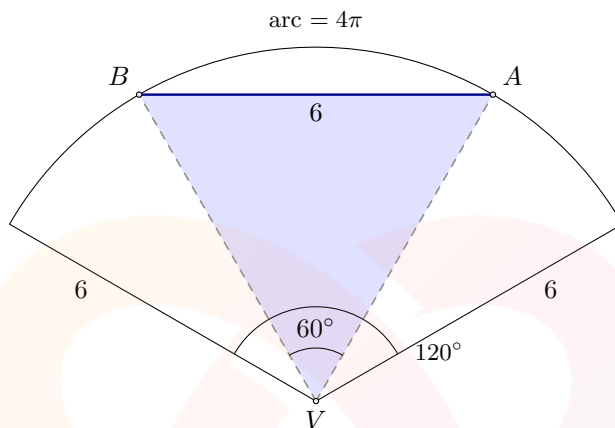
Question 14. Answer **D**.

Locker n is changed once for each distinct prime factor of n , so it ends up open exactly when n has an odd number of distinct prime factors. Numbers with one distinct prime factor are the prime powers: the 17 primes below 60 together with 4, 8, 16, 32, 9, 27, 25, 49, which is 25 lockers. Numbers up to 60 with three distinct prime factors are 30, 42 and 60, and four would need at least $2 \times 3 \times 5 \times 7 = 210$. The total is 28. The trap 7 is the answer to the classic version in which every divisor (not just every prime) has a student, leaving the perfect squares open; 25 forgets the numbers with three prime factors.

Question 15. Answer **E**.

Take any order of draws with k reds and $5 - k$ blues. The denominators are always 2, 3, 4, 5, 6, and the numerators for the reds are $1, 2, \dots, k$ and for the blues $1, 2, \dots, 5 - k$, whatever the order. So each such order has probability $\frac{k!(5-k)!}{6!}$, and there are $\binom{5}{k}$ orders, giving $\frac{5!}{6!} = \frac{1}{6}$ for every k from 0 to 5. At least 5 red counters at the end means at least 4 red draws, so the probability is $\frac{2}{6} = \frac{1}{3}$. The trap $\frac{3}{16}$ treats the draws as independent with probability $\frac{1}{2}$ each, but every red draw makes the next red draw more likely.

Question 16. Answer **F**.



In the net shown, the shortest route is the segment AB.

Cut the hat along a slant line and unroll it into a sector of radius 6 whose arc is the rim, of length 4π . The full circle of radius 6 has circumference 12π , so the sector angle is $\frac{1}{3} \times 360^\circ = 120^\circ$. Halfway round the rim corresponds to half of this, so in the net A and B are 60° apart as seen from the vertex V. The shortest route is the straight segment AB in the net, and triangle VAB has $VA = VB = 6$ with a 60° angle between them, so it is equilateral and $AB = 6$. This beats walking round the rim ($2\pi \approx 6.28$). The 4 cm diameter is not available because the hat has no base, and $6\sqrt{3}$ uses the full 120° .

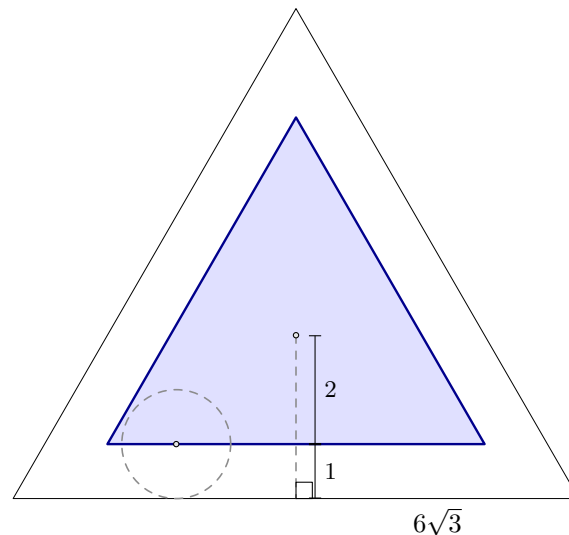
Question 17. Answer **D**.

We need $\frac{1}{2} \oplus \frac{1}{3} \oplus \dots \oplus \frac{1}{9}$. Working forwards, $\frac{1}{2} \oplus \frac{1}{3} = \frac{5}{7}$, then $\frac{5}{7} \oplus \frac{1}{4} = \frac{9}{11}$, then $\frac{9}{11} \oplus \frac{1}{5} = \frac{14}{16}$: the results are $\frac{T-1}{T+1}$ with $T = 6, 10, 15, \dots$ the triangular numbers. The reason is the identity

$$\frac{1+u \oplus v}{1-u \oplus v} = \frac{1+u+v+uv}{1-u-v+uv} = \frac{1+u}{1-u} \times \frac{1+v}{1-v},$$

so the quantity $\frac{1+w}{1-w}$ multiplies. For $w = \frac{1}{k}$ it equals $\frac{k+1}{k-1}$, and the product for $k = 2, \dots, 9$ telescopes to $\frac{9 \times 10}{1 \times 2} = 45$. Solving $\frac{1+w}{1-w} = 45$ gives $w = \frac{44}{46} = \frac{22}{23}$. The option $\frac{44}{45}$ comes from stopping at $T-1$ over T , and 1 from expecting the speeds to reach light speed.

Question 18. Answer **B**.



By symmetry we may consider the tile containing the centre. The coin avoids the lines when its centre is more than 1 cm from all three sides of the tile. These points form a smaller equilateral triangle with the same incentre and sides parallel to the tile's. The tile's inscribed circle has radius $\frac{6\sqrt{3}}{2\sqrt{3}} = 3$, so the small triangle has inradius $3 - 1 = 2$. The triangles are similar with length ratio $\frac{2}{3}$, so the area ratio, which is the probability, is $\frac{4}{9}$. The trap $\frac{2}{3}$ is the length ratio.

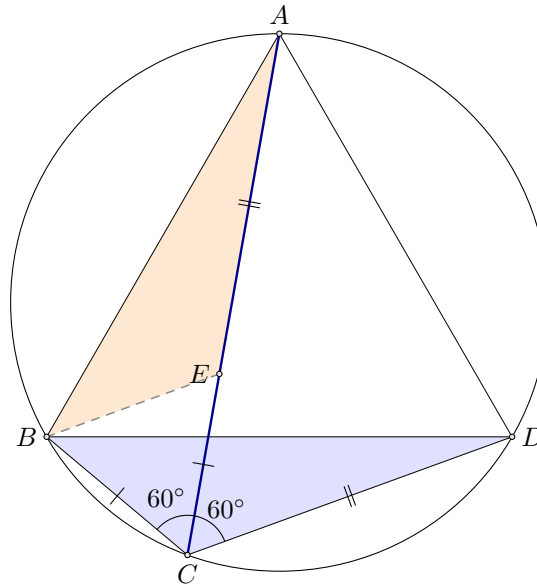
Question 19. Answer **A**.

The lights next coincide after the lowest common multiple of a and b seconds, so we need $\text{lcm}(a, b) = 72 = 2^3 \times 3^2$. Both a and b divide 72. For the prime 2, the pair of exponents must have larger value 3: of the 4×4 pairs from $\{0, 1, 2, 3\}$, the 3×3 with both below 3 fail, leaving 7. For the prime 3, similarly $3 \times 3 - 2 \times 2 = 5$. That gives 35 ordered pairs, including $a = b = 72$. Removing that one and halving, the number with $a < b$ is 17. The trap 35 forgets $a < b$, and 18 forgets to remove the pair $(72, 72)$.

Question 20. Answer **B**.

Each match eliminates one player, so exactly 15 matches are played. All $\binom{16}{2} = 120$ pairs of players are equally likely to be a pair who meet, by symmetry, and the expected number of pairs who meet is 15, so each pair meets with probability $\frac{15}{120} = \frac{1}{8}$. As a check, fix Ann's slot: Bea is in the same first-round match with probability $\frac{1}{15}$ (they meet for certain), in the same group of four but not that match with probability $\frac{2}{15}$ (they meet with probability $\frac{1}{4}$), in the same half but not that group with probability $\frac{4}{15}$ (probability $\frac{1}{16}$), and in the other half with probability $\frac{8}{15}$ (probability $\frac{1}{64}$), totalling $\frac{1}{15} (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = \frac{1}{8}$. The trap $\frac{1}{15}$ counts only round 1.

Question 21. Answer **E**.



Angles in the same segment give $\angle ACB = \angle ADB = 60^\circ$ and $\angle ACD = \angle ABD = 60^\circ$, so the area is $\frac{1}{2} AC \cdot CB \sin 60^\circ + \frac{1}{2} AC \cdot CD \sin 60^\circ = \frac{5\sqrt{3}}{2}(CB + CD)$. To find $CB + CD$, mark E on CA with $CE = CB$; triangle BCE has a 60° angle between equal sides, so it is equilateral. Then triangles BEA and BCD are congruent ($BE = BC$, $BA = BD$, and $\angle EBA = 60^\circ - \angle DBE = \angle CBD$), so $EA = CD$ and $CB + CD = CE + EA = CA = 10$. The area is $25\sqrt{3} \text{ m}^2$, the same for every position of C . A quick check: when C is the midpoint of arc BD , AC is a diameter and the bed is a kite with diagonals 10 and $5\sqrt{3}$, area $25\sqrt{3}$. The option $\frac{75\sqrt{3}}{4}$ is the area of triangle ABD alone in that case.

Question 22. Answer C.

Imagine she always makes all 6 tosses; she wins exactly when the sequence contains two consecutive heads. Let a_n be the number of sequences of n tosses with no two consecutive heads. Such a sequence either ends in a tail after a good sequence of length $n - 1$, or ends in tail then head after a good sequence of length $n - 2$, so $a_n = a_{n-1} + a_{n-2}$. With $a_1 = 2$ and $a_2 = 3$ we get 5, 8, 13, 21, so $a_6 = 21$. The probability of no prize is $\frac{21}{64}$, and of a prize $\frac{43}{64}$. The trap $\frac{781}{1024} = 1 - \left(\frac{3}{4}\right)^5$ treats the five overlapping pairs of tosses as independent.

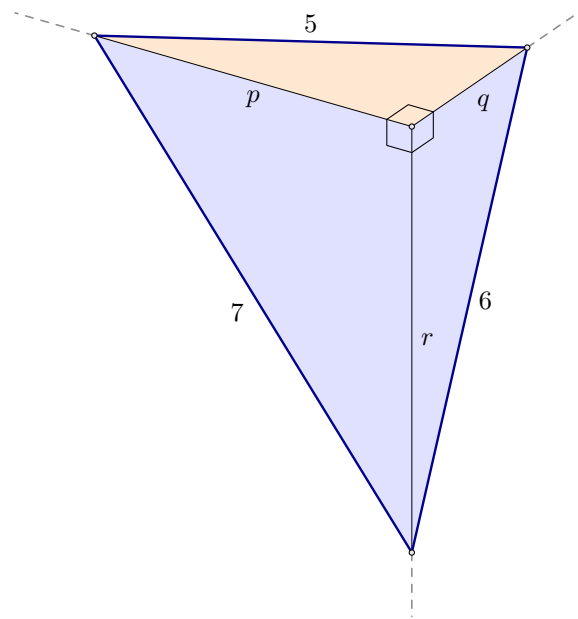
Question 23. Answer A.

The counter lands on square 4 exactly when the first few rolls add up to 4, and these events for different numbers of rolls cannot overlap. The ordered ways to make 4 are: 4 (1 way with one roll); 1 + 3, 3 + 1, 2 + 2 (3 ways with two rolls); 1 + 1 + 2 and its rearrangements (3 ways with three rolls); and 1 + 1 + 1 + 1. So the probability is

$$\frac{1}{6} + \frac{3}{36} + \frac{3}{216} + \frac{1}{1296} = \frac{216 + 108 + 18 + 1}{1296} = \frac{343}{1296} = \frac{1}{6} \left(\frac{7}{6}\right)^3.$$

The trap $\frac{2}{7}$ is the long-run chance for a square far along the track (1 divided by the mean roll 3.5), and $\frac{49}{216}$ is the answer for square 3.

Question 24. Answer F.

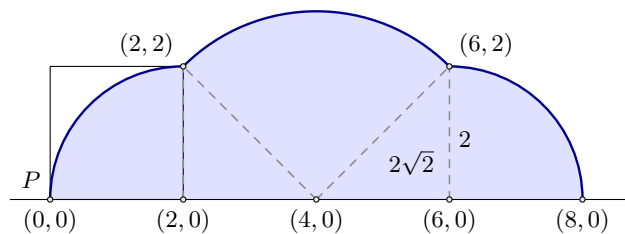


The piece is a pyramid whose three edges at the corner, p , q and r , are mutually perpendicular. Pythagoras on its three right-angled faces gives $p^2 + q^2 = 25$, $q^2 + r^2 = 36$ and $r^2 + p^2 = 49$. Adding, $p^2 + q^2 + r^2 = 55$, so $r^2 = 30$, $p^2 = 19$ and $q^2 = 6$. Taking a right-angled face as base, the volume is $\frac{1}{3} \times \frac{1}{2}pq \times r = \frac{pqr}{6} = \frac{\sqrt{19 \times 6 \times 30}}{6} = \frac{6\sqrt{95}}{6} = \sqrt{95} \text{ cm}^3$. The option $6\sqrt{95}$ forgets the $\frac{1}{6}$, and $6\sqrt{6}$ is the area of the triangular face.

Question 25. Answer **A**.

The stack in a given column and row can be no taller than the smaller of the two required maxima, and taking every stack exactly that tall shows both views correctly (each column and each row contains a square where its maximum is reached, because some row and some column allow height 4). So M is the sum of these minima: $9 + 4 + 11 + 7 = 31$. For m , each of the eight required heights must be shown by some stack, and one stack of height h can serve a column and a row that both need h . Pair the column needing 4 with a row needing 4, the 2 with the 2, and the 1 with the 1. The column needing 3 and the other row needing 4 remain, and need their own stacks (a 3 in a row allowing it, and a 4 in the column of height 4). So $m = 4 + 2 + 1 + 3 + 4 = 14$, and $M - m = 17$. The option 21 takes $m = 10$ by looking at the front view only, and 20 uses $m = 11$ from the side view only.

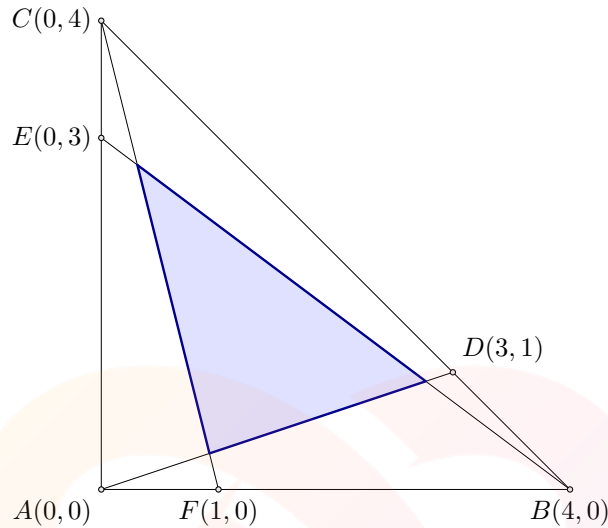
Question 26. Answer **B**.



Put P at the origin with the crate occupying $0 \leq x \leq 2$. In the first tip P turns about $(2,0)$ through a quarter circle of radius 2, reaching $(2,2)$. In the second it turns about $(4,0)$ through a quarter circle of radius $2\sqrt{2}$, reaching $(6,2)$. In the third it turns about $(6,0)$ through a quarter circle of radius 2, landing at $(8,0)$; in the fourth tip P is the pivot and does not move. The region under the

curve consists of two quarter discs of radius 2 (area π each), the quarter disc of radius $2\sqrt{2}$ centred at $(4,0)$ between the radii to $(2,2)$ and $(6,2)$ (area 2π), and the two right-angled triangles with vertices $(2,0)$, $(4,0)$, $(2,2)$ and $(4,0)$, $(6,0)$, $(6,2)$ (area 2 each). The total is $4\pi + 4$. The option 4π forgets the triangles.

Question 27. Answer **D**.



Stretching or shearing the triangle in a fixed direction scales all areas by the same factor and keeps the ratios along lines, so we may take $A(0,0)$, $B(4,0)$, $C(0,4)$, with area 8. Then $D = (3,1)$, $E = (0,3)$ and $F = (1,0)$, and the paths are AD : $y = \frac{1}{3}x$, BE : $3x + 4y = 12$, and CF : $4x + y = 4$. Their pairwise intersections are $(\frac{36}{13}, \frac{12}{13})$, $(\frac{4}{13}, \frac{36}{13})$ and $(\frac{12}{13}, \frac{4}{13})$. The shoelace formula gives area $\frac{1}{2} \times \frac{36 \times 32 - 4 \times 8 - 12 \times 24}{169} = \frac{416}{169} = \frac{32}{13}$, which is $\frac{4}{13}$ of 8. The option $\frac{1}{7}$ is the well-known answer when the points are one third of the way along the sides.