

HU Collection

ESAT Mock Set B: Mathematics 2

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	A	B	A	B	B	E	D	D	F	A

Q	11	12	13	14	15	16	17	18	19	20
Ans	A	E	A	B	B	D	B	F	E	C

Q	21	22	23	24	25	26	27			
Ans	F	F	D	D	B	A	F			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer A.

After t years the numbers of atoms are proportional to $8 \cdot 2^{-t/1200}$ and $2^{-t/1800}$, so the ratio of P to Q is

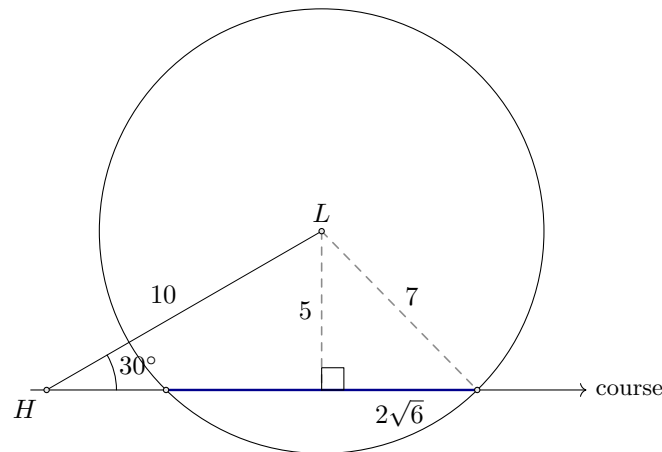
$$8 \cdot 2^{-t/1200+t/1800} = 8 \cdot 2^{-t/3600}.$$

The ratio itself halves every 3600 years. Setting $8 \cdot 2^{-t/3600} = \frac{1}{4}$ gives $2^{t/3600} = 32$, so $t = 5 \times 3600 = 18\,000$. Subtracting the half-lives (600 years per halving) gives 3000, adding the decay exponents instead of subtracting them gives 3600, following only one isotope gives 6000 or 9000, and reading the final ratio as 1 : 1 gives 10 800.

Question 2. Answer B.

Let the sides be a and $20 - a$. The side condition gives $7 \leq a \leq 13$. The diagonal condition is $a^2 + (20 - a)^2 \leq 225$, that is $400 - 2A \leq 225$, or $A \geq 87.5$. On $7 \leq a \leq 13$ the area $A = a(20 - a) = 100 - (a - 10)^2$ takes every value from 91 (at $a = 7$ or 13) to 100 (the square, whose diagonal $10\sqrt{2}$ is less than 15), so the diagonal condition is automatically met and the answer is $91 \leq A \leq 100$. Ignoring the side condition gives the lower bound 87.5; turning the diagonal condition into an upper bound via $2A \leq a^2 + b^2$ gives 112.5; multiplying the two minimum sides gives 49.

Question 3. Answer A.



The perpendicular distance from L to the course is $10 \sin 30^\circ = 5$ km. The course cuts the circle of radius 7 about L in a chord of half-length $\sqrt{49 - 25} = 2\sqrt{6}$, so the required distance is $4\sqrt{6}$ km. Both ends of the chord are ahead of H , because the foot of the perpendicular is $10 \cos 30^\circ = 5\sqrt{3} \approx 8.7$ km along the course, which exceeds $2\sqrt{6} \approx 4.9$. Equivalently, the cosine rule gives $s^2 - 10\sqrt{3}s + 51 = 0$ for the distance s sailed, and the roots differ by $\sqrt{300 - 204} = \sqrt{96}$. The sum of the roots, $10\sqrt{3}$, the larger root $5\sqrt{3} + 2\sqrt{6}$ and the half-chord $2\sqrt{6}$ are the usual slips.

Question 4. Answer B.

At $x = 0$ we have $y = 4 > 0$ for every k . For $0 < x \leq 3$, the condition $x^2 - 2kx + 4 > 0$ is equivalent to $2k < x + \frac{4}{x}$. On this interval $x + \frac{4}{x}$ has least value 4, at $x = 2$ (by differentiation, or since $x + \frac{4}{x} - 4 = \frac{(x-2)^2}{x}$). So the rules hold exactly when $2k < 4$, that is $k < 2$. Checking only the ends of the walkway gives $k < \frac{13}{6}$, but $k = 2.1$ fails at $x = 2$. Demanding a negative discriminant gives $-2 < k < 2$, which wrongly rejects negative k : then the vertex is to the left of the walkway and the truss is fine. At $k = 2$ the truss touches the walkway at $x = 2$.

Question 5. Answer B.

The n th term of the first progression is $S_n - S_{n-1} = 6n + 1$ (and the first term $S_1 = 7$ fits), so its terms are 7, 13, 19, ..., the numbers from 7 upwards that leave remainder 1 on division by 6. Similarly the second has n th term $10n - 7$, giving 3, 13, 23, ..., the numbers from 3 upwards ending in 3. A number is in both exactly when it is odd, leaves remainder 1 on division by 3 and remainder 3 on division by 5, that is, when it leaves remainder 13 on division by 30. These are 13, 43, ..., 973, which is 33 numbers. Comparing only terms in the same position ($6n + 1 = 10n - 7$) finds just one common value, 13.

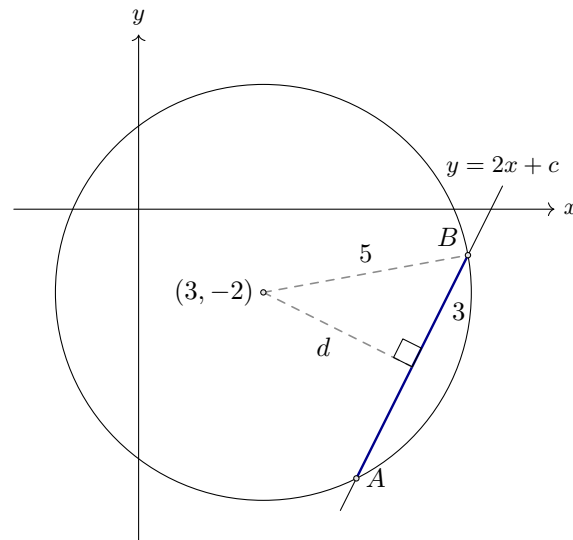
Question 6. Answer E.

The journey takes $\frac{300}{v}$ hours, so its cost is

$$J(v) = \frac{300}{v} \left(\frac{v^3}{1000} + 16 \right) = 0.3v^2 + \frac{4800}{v}.$$

Then $J'(v) = 0.6v - \frac{4800}{v^2}$, which is zero only when $v^3 = 8000$, that is $v = 20$, outside the permitted range. For $0 < v < 20$ we have $J'(v) < 0$, so J is decreasing throughout $0 < v \leq 15$ and the least cost is $J(15) = 67.5 + 320 = 387.5$. The unconstrained minimum $J(20) = 360$ is the trap; 320 counts only the fixed part of the cost.

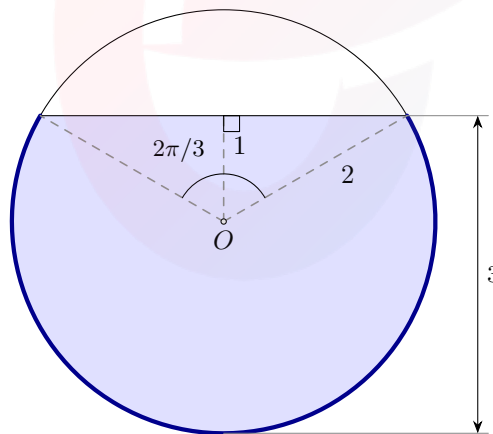
Question 7. Answer D.



The figure shows the extreme case, a chord of length exactly 6.

Completing the square, the circle has centre $(3, -2)$ and radius 5. A chord of length at least 6 has half-length at least 3, so its distance d from the centre satisfies $d \leq \sqrt{25 - 9} = 4$ (and then $d < 5$, so there really are two points). The distance from $(3, -2)$ to $2x - y + c = 0$ is $\frac{|6 + 2 + c|}{\sqrt{5}}$, so we need $|c + 8| \leq \sqrt{80}$. Since $8 < \sqrt{80} < 9$, the integers are those with $|c + 8| \leq 8$, namely $c = -16, -15, \dots, 0$, with sum -136 . A sign slip in the distance formula gives $+136$; using $d \leq 3$ gives -104 .

Question 8. Answer **D**.



The water surface is 1 m above the centre, so the dry part of the cross-section is a segment whose chord is 1 m from the centre; it subtends an angle θ with $\cos \frac{\theta}{2} = \frac{1}{2}$, so $\theta = \frac{2\pi}{3}$. The dry segment has area $\frac{1}{2} \cdot 4 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = \frac{4\pi}{3} - \sqrt{3}$, so the water has area $4\pi - \frac{4\pi}{3} + \sqrt{3} = \frac{8\pi}{3} + \sqrt{3}$. The wetted arc subtends $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$, so it has length $\frac{8\pi}{3}$. The hydraulic radius is $\frac{8\pi/3 + \sqrt{3}}{8\pi/3} = 1 + \frac{3\sqrt{3}}{8\pi}$.

Working with the minor segment as if the depth were 1 m gives $1 - \frac{3\sqrt{3}}{4\pi}$; including the free surface in the perimeter gives the fraction with $6\sqrt{3}$ in the denominator.

Question 9. Answer **F**.

The perpendicular bisector of AB passes through $(7, 6)$ with gradient -2 , and that of AC passes through $(4, 2)$ with gradient 3 ; they meet at the centre $(6, 8)$. The radius squared is $(1 - 6)^2 + (3 - 8)^2 = 50$. Since OT is tangent, the radius to T is perpendicular to OT , so by Pythagoras $OT^2 = (6^2 + 8^2) - 50 = 50$ and $OT = 5\sqrt{2}$. (Equivalently, the track is $x^2 + y^2 - 12x - 16y + 50 = 0$ and OT^2 is the constant term.) The distance to the centre is 10 , and the shortest distance from O to the track is $10 - 5\sqrt{2}$; adding instead of subtracting r^2 gives $5\sqrt{6}$.

Question 10. Answer **A**.

First $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$, and by the binomial expansion

$$(5 + 2\sqrt{6})^3 = 125 + 3 \cdot 25 \cdot 2\sqrt{6} + 3 \cdot 5 \cdot 24 + 48\sqrt{6} = 485 + 198\sqrt{6}.$$

Likewise $(5 - 2\sqrt{6})^3 = 485 - 198\sqrt{6}$, so the two cubes add to the integer 970 . Since $(5 - 2\sqrt{6})(5 + 2\sqrt{6}) = 1$, we have $0 < 5 - 2\sqrt{6} < \frac{1}{9}$, so $(5 - 2\sqrt{6})^3$ is a tiny positive number and $(\sqrt{2} + \sqrt{3})^6 = 970 - (5 - 2\sqrt{6})^3$ lies just below 970 . The answer is 969 . Estimating with $\sqrt{6} \approx 2.45$ gives about 970.1 and the wrong answer 970 .

Question 11. Answer **A**.

Since $y > 0$, $y = x^{3/2}$. Taking logarithms of $x^y = y^x$ gives $y \log x = x \log y = \frac{3}{2}x \log x$, so

$$\log x \left(x^{3/2} - \frac{3}{2}x \right) = 0.$$

If $\log x = 0$ then $x = y = 1$, which is excluded. Otherwise $x^{1/2} = \frac{3}{2}$, so $x = \frac{9}{4}$ and $y = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$, giving $x + y = \frac{45}{8}$. Check: both x^y and y^x equal $\left(\frac{3}{2}\right)^{27/4}$. Inverting the fraction when solving $x^{1/2} = \frac{3}{2}$ leads to $\frac{20}{27}$, and forgetting the condition $x \neq y$ suggests 2 .

Question 12. Answer **E**.

If the room temperature is R , the excesses $92 - R$, $56 - R$, $38 - R$ form a geometric progression, so $(56 - R)^2 = (92 - R)(38 - R)$. The R^2 terms cancel, leaving $3136 - 112R = 3496 - 130R$, so $R = 20$. The excesses are 72 , 36 , 18 : the excess halves every 10 minutes. A reading of 22.25°C is an excess of $2.25 = \frac{72}{32}$, which is five halvings after $09:00$, so the time is $09:50$. Assuming a room temperature of 0°C gives nonsense; measuring five halvings from the $09:20$ reading gives $10:10$, and confusing the 30 minutes after the last reading with a clock time gives $09:30$.

Question 13. Answer **A**.

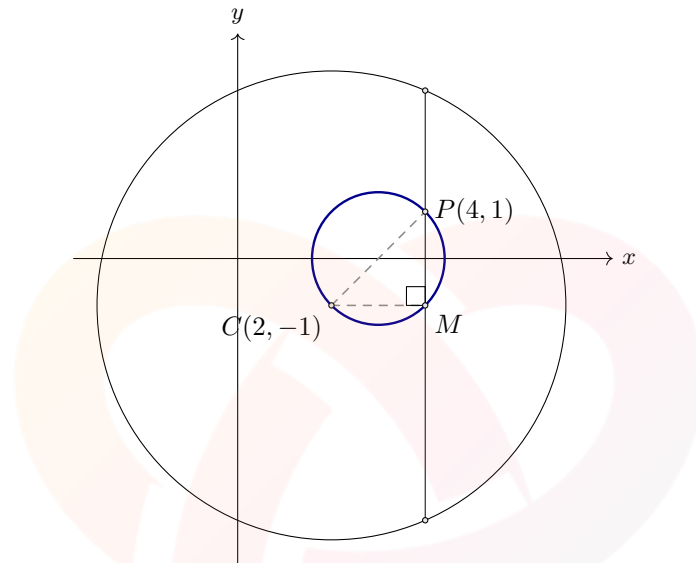
Let $a = 9^{\sin^2 x}$. Because $\sin^2 x + \cos^2 x = 1$, the second term is $9^{1-\sin^2 x} = \frac{9}{a}$, so $a + \frac{9}{a} = 4\sqrt{3}$, that is $a^2 - 4\sqrt{3}a + 9 = 0$, giving $a = \sqrt{3}$ or $a = 3\sqrt{3}$. These mean $\sin^2 x = \frac{1}{4}$ or $\sin^2 x = \frac{3}{4}$, so $\sin x \in \left\{ \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2} \right\}$. Each of these four values of $\sin x$ is taken exactly twice in $0 \leq x < 2\pi$, giving 8 solutions. Keeping only one root of the quadratic gives 4 .

Question 14. Answer **B**.

The point $(0, 2)$ lies on $y = \sqrt{x+4}$, so $x = 0$ is always a solution. Squaring gives $x + 4 = m^2x^2 + 4mx + 4$, so $x(m^2x + 4m - 1) = 0$. For $m = 0$ only $x = 0$ remains. For $m \neq 0$ the

other candidate is $x = \frac{1-4m}{m^2}$, and it is a genuine solution exactly when $mx + 2 \geq 0$ (then $x + 4 = (mx + 2)^2 \geq 0$ automatically). Here $mx + 2 = \frac{1-2m}{m}$, which is non-negative exactly when $0 < m \leq \frac{1}{2}$. The candidate differs from 0 unless $m = \frac{1}{4}$, when the line is the tangent at $(0, 2)$. At $m = \frac{1}{2}$ the second solution is the end point $x = -4$, which counts. Accepting every root of the squared equation gives $m \neq 0, \frac{1}{4}$; forgetting the end point gives the strict inequality.

Question 15. Answer **B**.

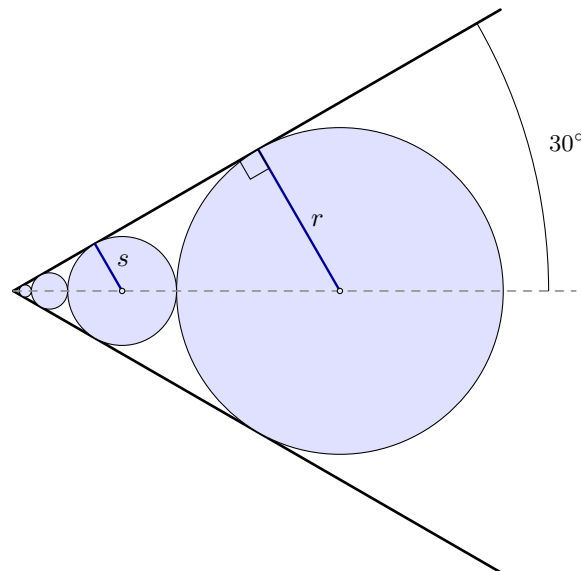


The boundary has centre $C(2, -1)$ and radius 5, and $CP = \sqrt{4+4} = 2\sqrt{2} < 5$, so P is inside. The line from the centre to the midpoint M of a chord is perpendicular to the chord, so angle CMP is a right angle (or M is C or P itself). Hence M lies on the circle with diameter CP , and every point of that circle arises from some chord through P . Its radius is $\sqrt{2}$, so the area is 2π . Using CP as the radius gives 8π .

Question 16. Answer **D**.

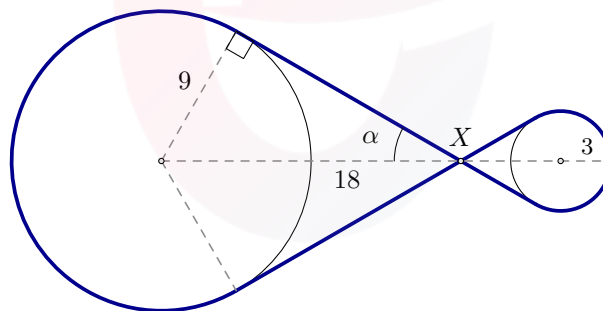
Integrating, $V = 100 + 4t^{3/2} - \frac{t^2}{2} - 8t$. Writing $u = \sqrt{t}$, the rate is $-(u^2 - 6u + 8) = -(u - 2)(u - 4)$, so V decreases until $t = 4$, increases until $t = 16$ and then decreases again. The candidates for the least value are the local minimum $V(4) = 100 + 32 - 8 - 32 = 92$ and the end point $V(25) = 100 + 500 - 312.5 - 200 = 87.5$. So the least volume is 87.5, reached at the end of the period, not at the stationary point. (Also $V(16) = 100$, so the greatest volume occurs twice.) Stopping at the stationary point gives 92.

Question 17. Answer **B**.



A disc of radius r touching both walls has its centre on the bisector of the corner, at distance $\frac{r}{\sin 30^\circ} = 2r$ from the corner. If the next disc has radius s , then the distance from the corner to the first centre is $2s + s + r$, so $2r = 3s + r$ and $s = \frac{r}{3}$. The areas form a geometric progression with first term 81π and common ratio $\frac{1}{9}$, so the total is $\frac{81\pi}{1 - 1/9} = \frac{729\pi}{8}$. Using the radius ratio $\frac{1}{3}$ for the areas gives $\frac{243\pi}{2}$; taking the radii to halve gives 108π .

Question 18. Answer **F**.



The crossing point X divides the line of centres in the ratio of the radii, so it is 18 cm from the larger centre and 6 cm from the smaller. Each straight section makes an angle α with the line of centres where $\sin \alpha = \frac{9}{18} = \frac{1}{2}$, so $\alpha = \frac{\pi}{6}$, and its length is $\sqrt{18^2 - 9^2} + \sqrt{6^2 - 3^2} = 9\sqrt{3} + 3\sqrt{3} = 12\sqrt{3}$. On each pulley the radii to the two points of tangency make an angle $2\left(\frac{\pi}{2} - \alpha\right) = \frac{2\pi}{3}$ on the side facing X , so the belt wraps through $2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$ on both pulleys. The arcs total $(9 + 3) \cdot \frac{4\pi}{3} = 16\pi$, and the length is $24\sqrt{3} + 16\pi$. Assuming a half-turn on each pulley gives 12π for the arcs; using the uncrossed tangents gives $12\sqrt{15}$ for the straight parts.

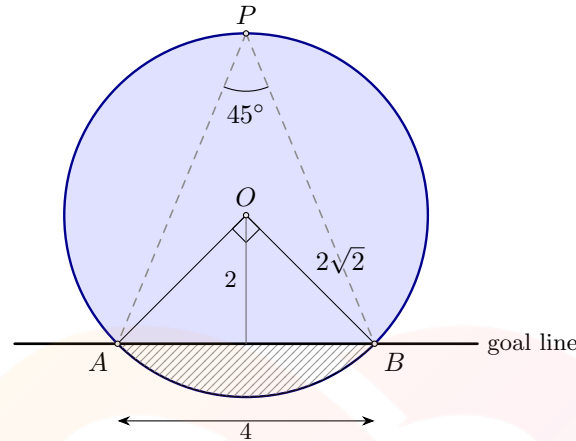
Question 19. Answer **E**.

The slant height of the cone is 6, so its base radius r and height h satisfy $r^2 + h^2 = 36$ and

$$V = \frac{\pi}{3} (36 - h^2) h = \frac{\pi}{3} (36h - h^3).$$

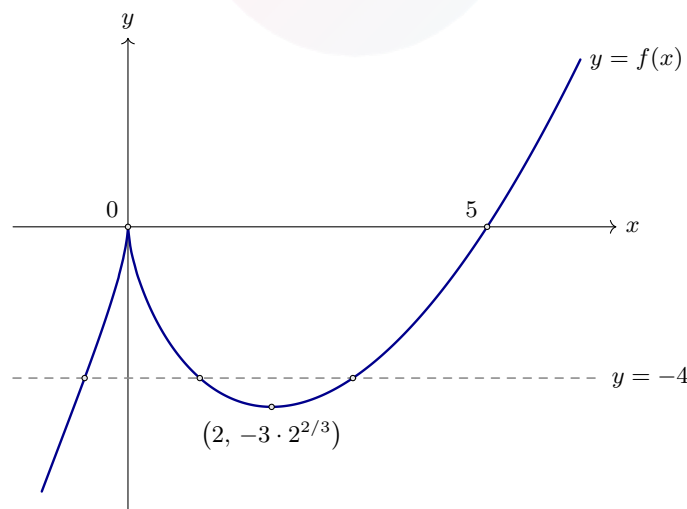
Then $\frac{dV}{dh} = \frac{\pi}{3} (36 - 3h^2)$, which is zero at $h = 2\sqrt{3}$, a maximum. There $r^2 = 24$, so $r = 2\sqrt{6}$. The arc of the sector becomes the base circumference: $6\theta = 2\pi \cdot 2\sqrt{6}$, giving $\theta = \frac{2\pi\sqrt{6}}{3}$ (about 294°). Swapping r and h gives $\frac{2\pi\sqrt{3}}{3}$; assuming $r = h$ at the optimum gives $\pi\sqrt{2}$.

Question 20. Answer **C**.



On the pitch side, the points with angle $APB = 45^\circ$ form an arc of the circle through A and B in which AB subtends 90° at the centre O ; this circle has radius $\frac{4}{\sqrt{2}} = 2\sqrt{2}$, and O is 2 m from AB on the pitch side. A point on the pitch side has shooting angle at least 45° exactly when it is inside or on this circle. So the region is the disc (area 8π) minus the minor segment beyond the goal line, whose area is $\frac{1}{2} \cdot 8 \left(\frac{\pi}{2} - 1 \right) = 2\pi - 4$. The area is $8\pi - (2\pi - 4) = 6\pi + 4$. Taking the whole disc gives 8π , a sign slip with the segment gives $6\pi - 4$, and including the far side of the goal line doubles the answer.

Question 21. Answer **F**.

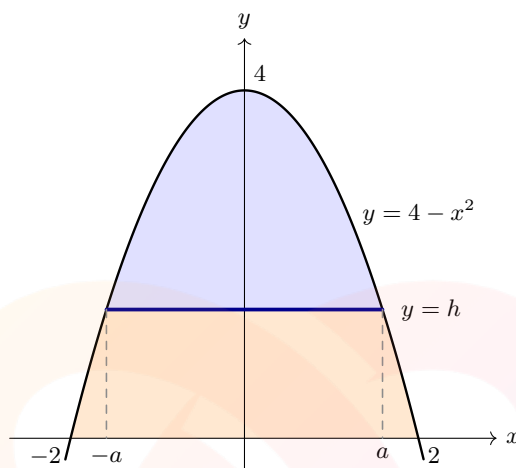


For $x \neq 0$, $f(x) = x^{5/3} - 5x^{2/3}$ gives

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5(x-2)}{3\sqrt[3]{x}}.$$

For $x < 0$ both $x - 2$ and $\sqrt[3]{x}$ are negative, so $f' > 0$ and f is increasing; for $0 < x < 2$, $f' < 0$; for $x > 2$, $f' > 0$. So II is false. Near $x = 0$ we have $x^{2/3} > 0$ and $x - 5 < 0$, so $f(x) < 0 = f(0)$: there is a local maximum at 0 (a cusp, not a stationary point), and I is true. The local minimum is $f(2) = -3 \cdot 2^{2/3}$, and $2^{2/3} > \frac{4}{3}$ because $4 > \frac{64}{27}$, so $f(2) < -4$. As f rises from $-\infty$ to 0 on $x < 0$, falls to $f(2)$, then rises without bound, the line $y = -4$ is crossed three times, so III is true. The trap is to look only for stationary points and miss the cusp.

Question 22. Answer **F**.



The part above the bar lies between $x = \pm a$, where $a = \sqrt{4 - h}$, and has area

$$\int_{-a}^a (a^2 - x^2) dx = \frac{4}{3}a^3 = \frac{4}{3}(4 - h)^{3/2}.$$

With $h = 0$ the whole window has area $\frac{32}{3}$, so we need $\frac{4}{3}(4 - h)^{3/2} = \frac{16}{3}$, that is $(4 - h)^{3/2} = 4$, so $4 - h = 4^{2/3} = 2\sqrt[3]{2}$ and $h = 4 - 2\sqrt[3]{2} \approx 1.48$. The value $2\sqrt[3]{2}$ is the depth of the bar below the top, not its height; assuming the area above the bar is proportional to its depth gives $h = 2$, and assuming it is proportional to the square of the depth gives $4 - 2\sqrt{2}$.

Question 23. Answer **D**.

We have $f'(x) = k(x - 1)(x - 3)$, and $f(1) - f(3) = -\int_1^3 k(x - 1)(x - 3) dx = \frac{4k}{3} = 4$, so $k = 3$, and with $f(3) = 0$ this gives $f(x) = x^3 - 6x^2 + 9x = x(x - 3)^2$. So $f \geq 0$ for $x \geq 0$; f rises from 0 to 4 on $[0, 1]$, falls to 0 at 3 and then rises, with $f(4) = 4$. For $f(|x|) = c$, count the solutions with $x > 0$, double them, and add $x = 0$ if $c = 0$. I: three positive solutions, so 6. II: $f = 1$ has three solutions and $f = -1$ has one (negative) solution, so 4. III: $f(|x|) = -3$ is impossible and $f(|x|) = 3$ gives 6. IV: the positive solutions are $x = 1$ and $x = 4$, so 4. V: $f = 4$ at $x = 1$ and $x = 4$, and $f = -4$ once for $x < 0$, so 3. VI: $x = 0, \pm 3$, so 3. Exactly two equations (II and IV) have four solutions.

Question 24. Answer **D**.

Write T_2 and T_4 for the two estimates and I for the integral. All three depend linearly on f , so it is enough to compare them for $f = 1, x, x^2, x^3$. For 1 and x the trapezium rule is exact. For x^2 : $I = \frac{64}{3}$, $T_2 = 24$, $T_4 = 22$; for x^3 : $I = 64$, $T_2 = 80$, $T_4 = 68$. In both cases $T_2 - I = 4(T_4 - I)$: halving the strip width divides the error by 4. Hence $I = \frac{4T_4 - T_2}{3} = \frac{74 - 20}{3} = 18$ (this is Simpson's rule). Such a cable exists, for example $f(x) = 3.5 + 0.75(x - 2)^2$. Taking T_4 as exact gives 18.5; assuming the error halves gives 17.75; assuming it is unchanged gives 17.

Question 25. Answer **B**.

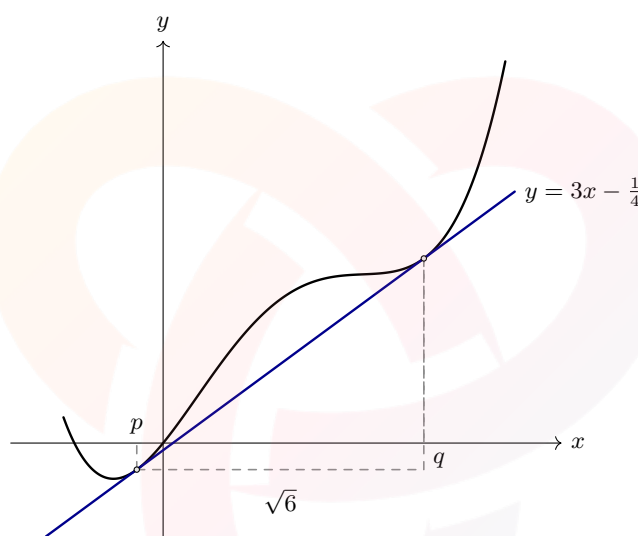
The displacement s from O satisfies $\frac{ds}{dt} = v$, so integrating, s is a polynomial of degree 4 with t^4 coefficient 1. It vanishes at $t = 0, 1, 2, 3$, so

$$s = t(t-1)(t-2)(t-3) = (t^2 - 3t)(t^2 - 3t + 2).$$

With $w = t^2 - 3t$, which runs over $-\frac{9}{4} \leq w \leq 0$ for $0 \leq t \leq 3$, we get $s = w(w+2) = (w+1)^2 - 1$.

This ranges from -1 (at $w = -1$, that is $t = \frac{3 \pm \sqrt{5}}{2}$) up to $\frac{25}{16} - 1 = \frac{9}{16}$ (at $w = -\frac{9}{4}$, that is $t = \frac{3}{2}$). The greatest distance is 1 m. The symmetry about $t = \frac{3}{2}$ tempts many to stop at the local maximum $\frac{9}{16}$, but the particle goes further on the negative side.

Question 26. Answer **A**.



If the line $y = mx + c$ touches the curve at $x = p$ and $x = q$ with $p \neq q$, then $x^4 - 4x^3 + 3x^2 + (5-m)x - c$ has double roots at p and q , so it equals $(x^2 - Sx + P)^2$ with $S = p + q$, $P = pq$. Expanding, $(x^2 - Sx + P)^2 = x^4 - 2Sx^3 + (S^2 + 2P)x^2 - 2SPx + P^2$. Comparing coefficients: $2S = 4$, so $S = 2$; $S^2 + 2P = 3$, so $P = -\frac{1}{2}$; then $5 - m = -2SP = 2$, so $m = 3$, and $c = -P^2 = -\frac{1}{4}$. The points of contact satisfy $x^2 - 2x - \frac{1}{2} = 0$, so they differ in x by $\sqrt{4 + 2} = \sqrt{6}$, and since the rod has gradient 3 their distance apart is $\sqrt{6} \cdot \sqrt{1 + 3^2} = \sqrt{60} = 2\sqrt{15}$. Stopping at the horizontal separation gives $\sqrt{6}$.

Question 27. Answer **F**.

The height is $H = \sum_{k=1}^{10000} k^{-1/2}$. Comparing rectangles with the area under $y = x^{-1/2}$ gives $2\sqrt{10001} - 2 < H < 1 + \int_1^{10000} x^{-1/2} dx = 199$, so $198 < H < 199$; this fixes the integer part but not the rounding. Now apply the trapezium rule to $\int_1^{10000} x^{-1/2} dx = 198$ with unit strips: the estimate is $H - \frac{1}{2} \left(1 + \frac{1}{100} \right) = H - 0.505$. Since $\frac{d^2}{dx^2} x^{-1/2} = \frac{3}{4} x^{-5/2} > 0$, the curve bends upwards and every trapezium lies above it, so the estimate exceeds the integral: $H - 0.505 > 198$. Hence $198.505 < H < 199$ and the height is 199 cm to the nearest centimetre (in fact $H \approx 198.54$). Quoting the integer part gives 198, and using $2\sqrt{10000}$ gives 200.