

HU Collection

ESAT Mock Set B: Mathematics 1

Worked Solutions

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Answer key

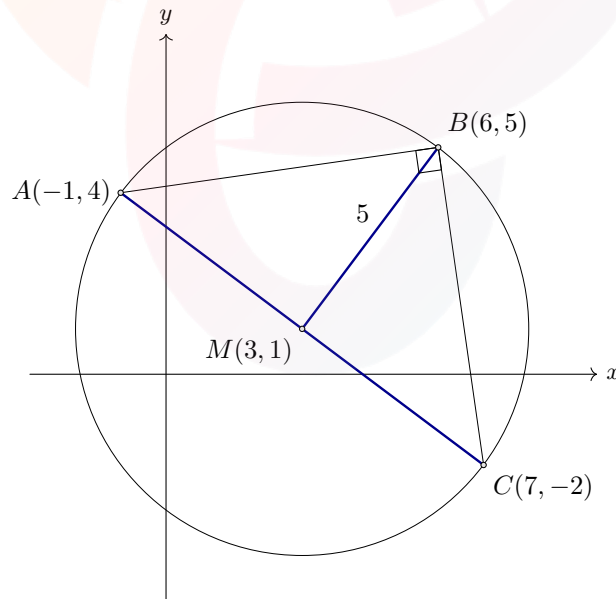
Q	1	2	3	4	5	6	7	8	9	10
Ans	E	A	E	B	B	B	A	E	C	F

Q	11	12	13	14	15	16	17	18	19	20
Ans	F	C	C	B	B	D	D	F	D	E

Q	21	22	23	24	25	26	27			
Ans	A	F	E	F	C	B	E			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer **E**.



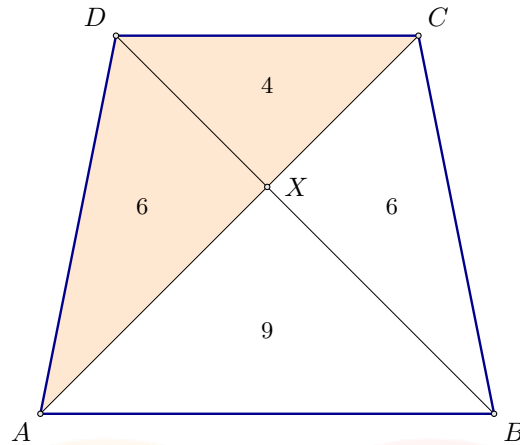
M is equidistant from all three beacons, so it is the centre of the circle through A , B and C . The gradient of AB is $\frac{1}{7}$ and the gradient of BC is -7 ; their product is -1 , so $\angle ABC = 90^\circ$. By the angle in a semicircle, AC is a diameter, so M is the midpoint $(3, 1)$ of AC and $MB = \frac{1}{2}AC = \frac{1}{2}\sqrt{8^2 + 6^2} = 5$ km. Solving the two perpendicular bisectors, $y = -7x + 22$ and $x - 7y = -4$, gives the same point, but the right angle saves the algebra. The option 10 km is the diameter AC itself.

Question 2. Answer **A**.

The original volume is $1800 \div 7.2 = 250 \text{ cm}^3$. If it contains $v \text{ cm}^3$ of P , then $9v + 6(250 - v) = 1800$, so $v = 100$: the ingot holds 900 g of P and 900 g of Q . Adding $x \text{ cm}^3$ of P gives $\frac{1800 + 9x}{250 + x} = 7.5$, so

$1.5x = 75$ and $x = 50$, which is 450 g. The new ingot contains 1350 g of P out of 2250 g, a fraction $\frac{3}{5}$. The trap $\frac{1}{2}$ is the new *volume* fraction (150 out of 300 cm^3), and $\frac{2}{5}$ is the original volume fraction.

Question 3. Answer **E**.

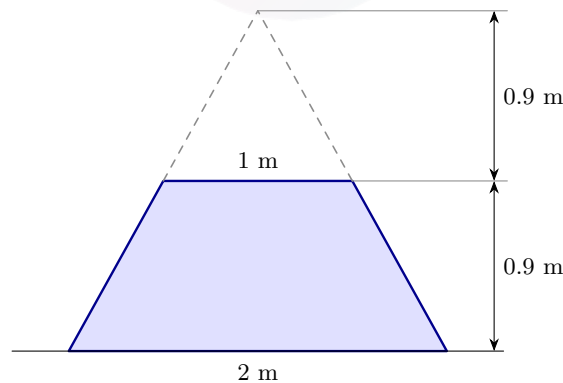


Triangles ABD and ABC share the base AB and have equal heights, so removing the common part AXB shows that BXC also has area 6. Triangles DXC and BXC have the same height from C , so $DX : XB = 4 : 6 = 2 : 3$. Triangles DXC and BXA are similar (alternate angles), with lengths in the ratio $2 : 3$, so AXB has area $4 \times \frac{9}{4} = 9$. The total is $4 + 6 + 6 + 9 = 25 \text{ m}^2$. Scaling the area linearly ($AXB = 6$, total 22) or assuming a fourth bed equal to DXC (total 20) are the traps.

Question 4. Answer **B**.

Two significant figures in 0.40 km means the nearest 0.01 km, so the length d satisfies $395 \leq d < 405$ metres. Truncation means the time t satisfies $50 \leq t < 51$ seconds. The upper bound of the speed is $405 \div 50 = 8.1 \text{ m/s}$, and $8.1 \times 3.6 = 29.16 \text{ km/h}$. Treating the time as rounded, with lower bound 49.5, gives $29.\dot{4}\dot{5}$; reading 0.40 as one significant figure (so $d < 450$) gives 32.4; and 8.1 is the answer in metres per second.

Question 5. Answer **B**.



The figure shows a side view of the plinth with the sloping faces extended to the apex.

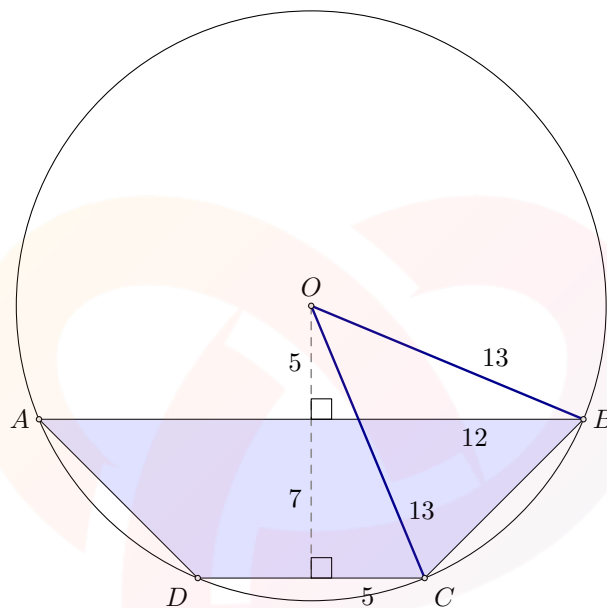
Extend the sloping faces to a complete pyramid. The side length halves, from 2 m to 1 m, over a height of 0.9 m, so the apex is 1.8 m above the ground and the missing top pyramid has height 0.9 m. The volume is $\frac{1}{3}(4)(1.8) - \frac{1}{3}(1)(0.9) = 2.4 - 0.3 = 2.1 \text{ m}^3$, a mass of 5040 kg. The total mass is 6240 kg, a weight of 62 400 N, spread over 4 m^2 : the pressure is $15\,600 \text{ Pa} = 15.6 \text{ kPa}$. Treating the

frustum as a prism whose cross-section is the mean of the two faces (16.5) or the square of side 1.5 m (15.15), or forgetting the statue (12.6), are the traps.

Question 6. Answer B.

First $x + y = 26$, and using midpoints $8(5) + 15x + 20(25) + 35y + 6(50) = 60 \times 24.5 = 1470$, so $15x + 35y = 630$. Hence $x = 14$ and $y = 12$. On a histogram, height is proportional to frequency density. The last class has density $6 \div 20 = 0.3$, drawn as 1.5 cm, so each unit of density is 5 cm. The class $10 \leq L < 20$ has density $14 \div 10 = 1.4$, so its bar is 7 cm tall. Scaling frequencies instead of densities gives 3.5 cm, and swapping x and y gives 6 cm.

Question 7. Answer A.



The perpendicular from the centre to a chord bisects the chord, and for both parallel chords this perpendicular lies along the axis of symmetry. So O is $\sqrt{13^2 - 12^2} = 5$ cm from AB and $\sqrt{13^2 - 5^2} = 12$ cm from CD . If O were inside the trapezium the height would be $5 + 12 = 17$ cm, giving the tempting 289. Since O is outside, both chords lie on the same side of O and the height is $12 - 5 = 7$ cm, so the area is $\frac{1}{2}(24 + 10)(7) = 119 \text{ cm}^2$.

Question 8. Answer E.

Every exponent is a multiple of 10, so compare the tenth roots: $2^{15} = 32768$, $3^{10} = 59049$, $5^6 = 15625$, $7^5 = 16807$ and $11^4 = 14641$. Hence $11^{40} < 5^{60} < 7^{50} < 2^{150} < 3^{100}$. Ordering by base, or by exponent, fails, and the three smallest values are close enough that they have to be computed rather than estimated.

Question 9. Answer C.

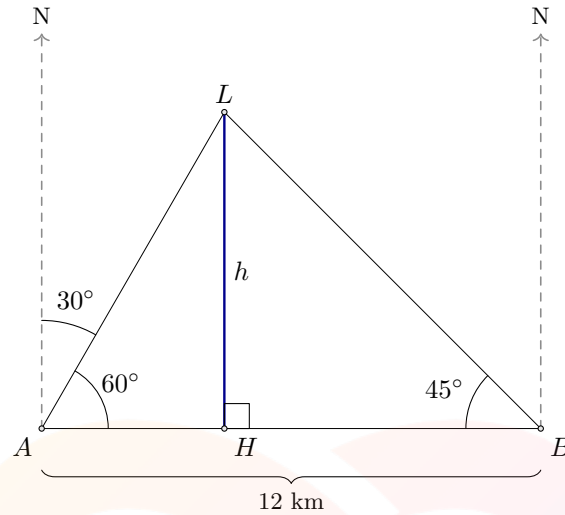
Over two equal distances s at 60 and 120 km/h the time is $\frac{s}{60} + \frac{s}{120} = \frac{s}{40}$, so the average speed for the second phase is $2s \div \frac{s}{40} = 80$ km/h (the harmonic mean), not 90. The two phases last equal times, so the overall average is the ordinary mean of 100 and 80, namely 90 km/h. Using 90 for the second phase gives 95; averaging the three speeds gives $93\frac{1}{3}$; treating all three stretches as equal distances gives $85\frac{5}{7}$.

Question 10. Answer F.

Per 10 000 people: 200 have the condition and $200 \times 0.9 \times 0.9 = 162$ of them test positive twice; 9800 do not, and $9800 \times 0.05 \times 0.05 = 24.5$ of them test positive twice. The required probability is

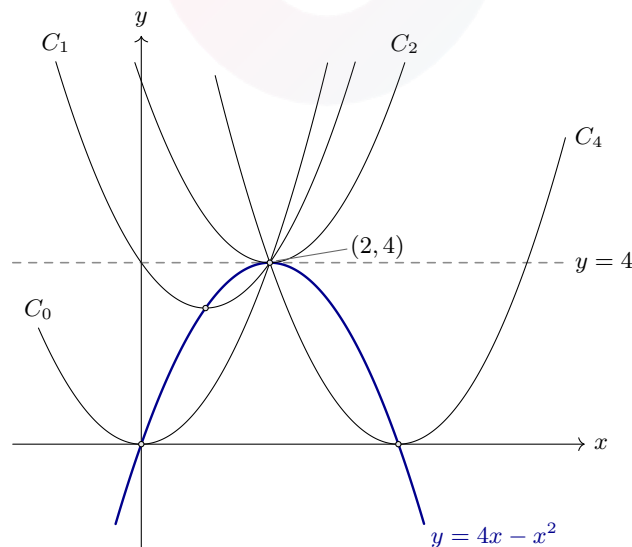
$\frac{162}{162 + 24.5} = \frac{324}{373}$. Stopping after one test gives $\frac{18}{67}$; forgetting the factor 0.98 gives $\frac{162}{187}$; and $\frac{81}{100}$ is the probability of two positives given the condition, the reverse conditional.

Question 11. Answer **F**.



The bearing 030° makes $\angle LAB = 90^\circ - 30^\circ = 60^\circ$, and from B the direction back to A is 270° , so $\angle LBA = 315^\circ - 270^\circ = 45^\circ$. Both angles are acute, so the foot H of the perpendicular from L lies between A and B . With $LH = h$, $AH = \frac{h}{\tan 60^\circ} = \frac{h}{\sqrt{3}}$ and $HB = h$, so $h \left(1 + \frac{1}{\sqrt{3}}\right) = 12$, giving $h = \frac{12\sqrt{3}}{\sqrt{3} + 1} = 6\sqrt{3}(\sqrt{3} - 1) = 18 - 6\sqrt{3}$ km. Using the bearing 30° itself as the angle at A gives $6\sqrt{3} - 6$; $12\sqrt{3} - 12$ is the length AL .

Question 12. Answer **C**.



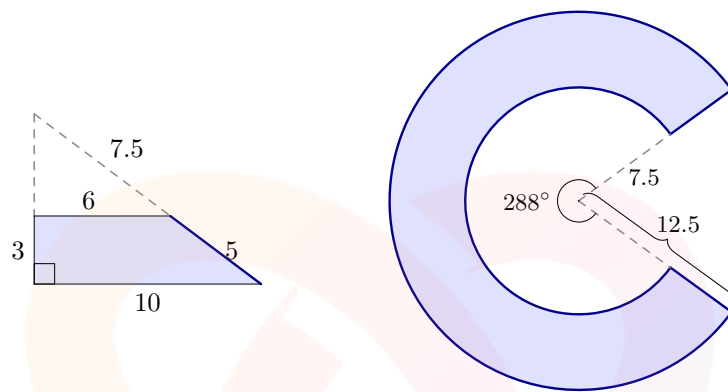
Write $y = x^2 + k(4 - 2x)$: at $x = 2$ we get $y = 4$ whatever the value of k , so every C_k passes through $(2, 4)$ and I is true. Completing the square, $y = (x - k)^2 + 4k - k^2$, so the turning point is $(k, 4k - k^2)$, which lies on $y = 4x - x^2$: II is true. Its y -coordinate is $4 - (k - 2)^2 \leq 4$, so III is true. But for

$0 < k < 4$ the minimum value $4k - k^2$ is positive; for example C_2 is $y = (x - 2)^2 + 4$, which never meets the x -axis. So IV is false.

Question 13. Answer **C**.

The product is a multiple of 10 exactly when at least one die shows 5 and at least one die shows an even number. For the complement: no 5 has probability $\frac{125}{216}$, no even number has probability $\frac{27}{216}$, and both together (every die in $\{1, 3\}$) has probability $\frac{8}{216}$. So the answer is $1 - \frac{125}{216} - \frac{27}{216} + \frac{8}{216} = \frac{72}{216} = \frac{1}{3}$. Forgetting to add back the overlap gives $\frac{8}{27}$. Multiplying $P(\text{some } 5) = \frac{91}{216}$ by $P(\text{some even}) = \frac{7}{8}$ gives $\frac{637}{1728}$, but these events are not independent, since a die showing 5 cannot also be even.

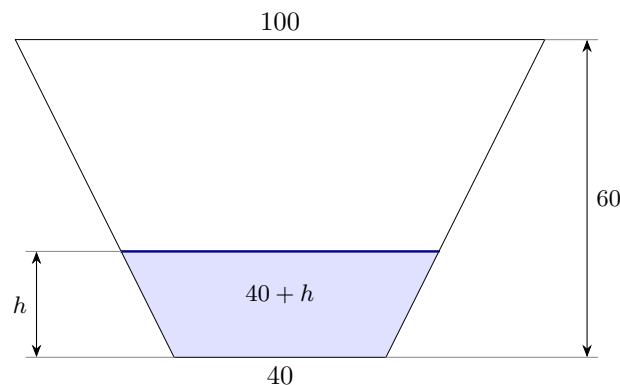
Question 14. Answer **B**.



In the figure, the left diagram is half of a vertical section through the axis of the completed cone and the right diagram is the flat piece of card.

The slant height of the frustum is $\sqrt{3^2 + 4^2} = 5$ cm. Complete the cone: its slant height ℓ satisfies $\frac{\ell}{10} = \frac{\ell - 5}{6}$, so $\ell = 12.5$, and the removed cone has slant height 7.5. These slant heights are the outer and inner radii of the annular sector. The outer arc becomes the larger rim, of length 20π , so the angle is $\frac{20\pi}{2\pi \times 12.5} \times 360^\circ = 288^\circ$ (the inner arc checks: $\frac{288}{360} \times 2\pi \times 7.5 = 12\pi$). Taking the inner radius to be the rim radius 6 cm, or using $\frac{6}{10}$ of 360° , are the traps.

Question 15. Answer **B**.



In 20 minutes 300 litres, that is $300\,000 \text{ cm}^3$, flows in; dividing by the length 300 cm, the water's cross-section has area 1000 cm^2 . The width increases by 60 cm over a depth of 60 cm, so at height h above the base the width is $40 + h$. The water's cross-section is a trapezium of area

$\frac{1}{2}(40 + 40 + h)h = 1000$, so $h^2 + 80h - 2000 = 0$, which factorises as $(h + 100)(h - 20) = 0$, and $h = 20$ cm. Treating the water as a rectangle of width 40, 70 or 100 cm gives 25, $14\frac{2}{7}$ or 10 cm.

Question 16. Answer **D**.

Let the smallest value be a , so the largest is $a + 40$, and the three values below 10 force $a \leq 7$. To maximise, take $a = 7$: the values are 7, 8, 9, 10, 17, 18, 19, 20, 27, 28, 29, 30, 45, 46, 47, with total 360. Note that raising a also raises the largest value, so both ends push the same way. To minimise, take $a = 1$: the values are 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 41, with total 272. So $G - S = \frac{360 - 272}{15} = \frac{88}{15}$. Fixing the extremes independently, for example keeping the largest value at 41 when maximising, leads to the other options.

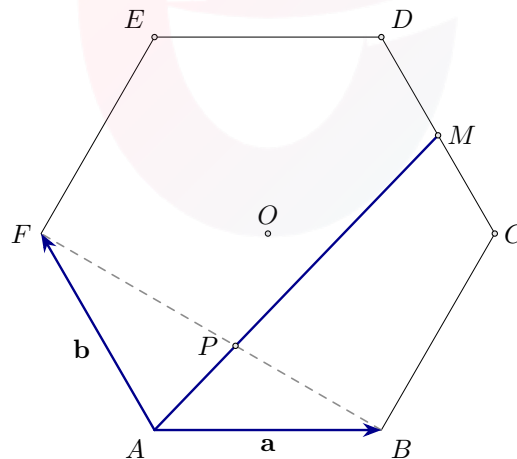
Question 17. Answer **D**.

With r red and b blue counters and $n = r + b$, the condition is $\frac{r(r-1) + b(b-1)}{n(n-1)} = \frac{1}{2}$, so $2r^2 + 2b^2 - 2n = n^2 - n = (r+b)^2 - (r+b)$. This simplifies to $(r-b)^2 = r+b$. So n must be a perfect square, and the only square from 30 to 40 is 36, giving $r-b = 6$, $r+b = 36$, so $r = 21$. The option 15 comes from $n = 25$, which is outside the allowed range.

Question 18. Answer **F**.

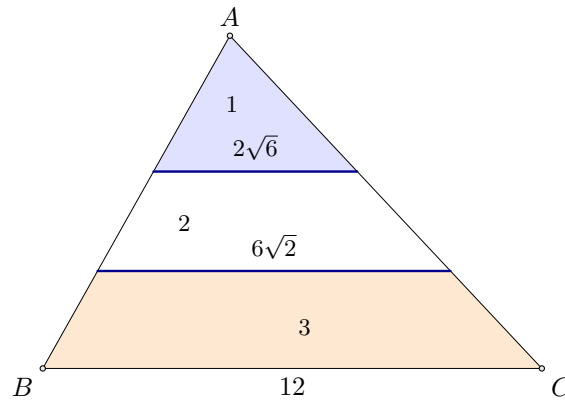
Count triples $a \leq b \leq c$ with $abc = 360 = 2^3 \times 3^2 \times 5$, listing by the smallest edge a . For $a = 1$: $bc = 360$, which has 24 divisors, giving 12 pairs. For $a = 2$: $bc = 180$ with $b \geq 2$, giving $9 - 1 = 8$. For $a = 3$: $bc = 120$ with $b \geq 3$, giving $8 - 2 = 6$. For $a = 4$: $bc = 90$ with $4 \leq b \leq 9$, so $b = 5, 6, 9$, giving 3. For $a = 5$: $bc = 72$ with $5 \leq b \leq 8$, so $b = 6, 8$, giving 2. For $a = 6$: $bc = 60$ with $b = 6$ only, giving 1. For $a \geq 7$ there is nothing, since $a^3 \leq 360$ and 7 does not divide 360. The total is $12 + 8 + 6 + 3 + 2 + 1 = 32$. The number of *ordered* triples is 180.

Question 19. Answer **D**.



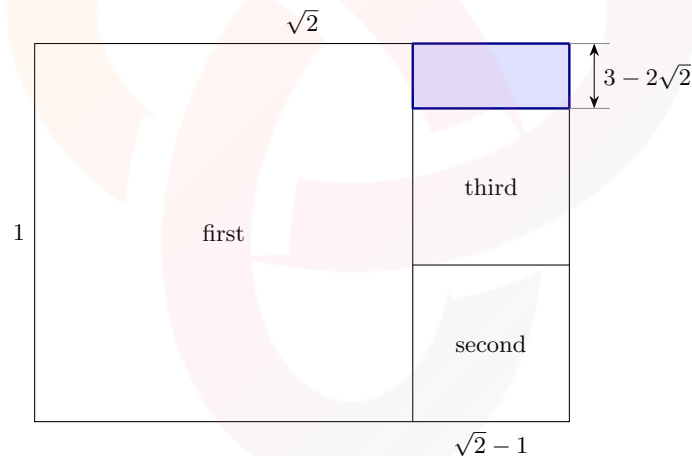
The hexagon is made of six equilateral triangles about its centre O , so $\overrightarrow{AO} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{BO} = \mathbf{a}$. Hence $\overrightarrow{AC} = 2\mathbf{a} + \mathbf{b}$; also $\overrightarrow{CD} = \overrightarrow{AF} = \mathbf{b}$, so $\overrightarrow{AM} = 2\mathbf{a} + \frac{3}{2}\mathbf{b}$. A point on AM is $t(2\mathbf{a} + \frac{3}{2}\mathbf{b})$ and a point on BF is $\mathbf{a} + u(\mathbf{b} - \mathbf{a}) = (1-u)\mathbf{a} + u\mathbf{b}$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $2t = 1-u$ and $\frac{3}{2}t = u$, so $t = \frac{2}{7}$ and $\overrightarrow{AP} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{b}$. The option $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$ also lies on BF but not on AM ; $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ is the midpoint of BF .

Question 20. Answer **E**.



Each fence cuts off at A a triangle similar to ABC . The nearer fence cuts off $\frac{1}{6}$ of the area and the further one cuts off $\frac{1}{6} + \frac{2}{6} = \frac{1}{2}$. Lengths scale as the square root of areas, so the fences have lengths $12\sqrt{\frac{1}{6}} = 2\sqrt{6}$ and $12\sqrt{\frac{1}{2}} = 6\sqrt{2}$, a total of $2\sqrt{6} + 6\sqrt{2}$ m. Taking lengths proportional to areas gives $2 + 6 = 8$; reading the ratio from BC instead of from A gives $6\sqrt{2} + 2\sqrt{30}$; equal areas would give $4\sqrt{3} + 4\sqrt{6}$.

Question 21. Answer **A**.



Take the sides as 1 and $\sqrt{2}$, area $\sqrt{2}$. The first cut leaves a 1 by $\sqrt{2} - 1$ strip. Since $\sqrt{2} - 1 < 1$, the second square has side $\sqrt{2} - 1$ and leaves $(\sqrt{2} - 1)$ by $(2 - \sqrt{2})$. Since $\sqrt{2} - 1 < 2 - \sqrt{2}$, the third square has side $\sqrt{2} - 1$ again and leaves $(\sqrt{2} - 1)$ by $(3 - 2\sqrt{2}) = (\sqrt{2} - 1)^2$. The remaining area is $(\sqrt{2} - 1)^3 = 5\sqrt{2} - 7$, and as a fraction of $\sqrt{2}$ this is $\frac{5\sqrt{2} - 7}{\sqrt{2}} = 5 - \frac{7}{\sqrt{2}} = \frac{10 - 7\sqrt{2}}{2}$. Forgetting to divide by the original area gives $5\sqrt{2} - 7$; $3 - 2\sqrt{2}$ is the fraction after only two cuts.

Question 22. Answer **F**.

With first two terms a and b , the terms are $a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, 8a + 13b, 13a + 21b, 21a + 34b$. So we need $21a + 34b = 2026$ with $a, b \geq 1$, that is, $2026 - 34b$ must be a positive multiple of 21. As $2026 = 21 \times 96 + 10$ and $34b = 21b + 13b$, this needs $13b - 10$ to be a multiple of 21. The smallest such b is 4 (since $52 - 10 = 42$), and then b can go up in steps of 21. The values $b = 4, 25, 46$ give $a = 90, 56, 22$, while $b = 67$ makes a negative. So there are 3 sequences.

Question 23. Answer **E**.

In order the integers are $a \leq b \leq c \leq 9 \leq e \leq f \leq g$ with the middle one equal to 9 and total 70. Since 12 is a mode it appears at least twice, and as $12 > 9$ it fills at least two of the places e, f, g .

Greatest range: take $e = f = 12$, with 12 appearing exactly twice, so every other value may appear only once; then a, b, c are at least 1, 2, 3 and $g = 70 - (1 + 2 + 3 + 9 + 24) = 31$, so $R = 30$. (Taking $a = b = c = 1$ would make 1 a mode too.) Least range: with $e = f = g = 12$, $a + b + c = 25$ with $c \leq 9$; no value other than 12 may appear three times, so $(7, 9, 9)$ fails and the best is $(8, 8, 9)$, giving $r = 12 - 8 = 4$. If 12 appears only twice, every other value appears once, so $c \leq 8$ and $a + b + c \leq 6 + 7 + 8 = 21$; this forces $e = f = 12$ and $g \geq 16$, and the range is at least 10. So $R - r = 26$.

Question 24. Answer **F**.

A regular n -gon has interior angle $180^\circ - \frac{360^\circ}{n}$, so the condition becomes $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ with $3 \leq a < b < c$. Since $\frac{3}{a} > \frac{1}{2}$, $a \leq 5$. For $a = 3$: $(b - 6)(c - 6) = 36$, giving $(7, 42), (8, 24), (9, 18), (10, 15)$. For $a = 4$: $(b - 4)(c - 4) = 16$, giving $(5, 20), (6, 12)$ (the factorisation 4×4 would make $b = c$). For $a = 5$: $\frac{1}{b} + \frac{1}{c} = \frac{3}{10}$ needs $b < \frac{20}{3}$, so $b = 6$, but then $c = \frac{15}{2}$. That makes 6 sets. Allowing repeated polygons would add $\{3, 12, 12\}, \{4, 8, 8\}, \{5, 5, 10\}, \{6, 6, 6\}$, giving the trap 10.

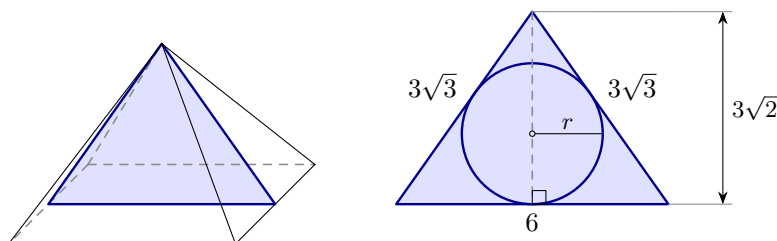
Question 25. Answer **C**.

For Ana to win in exactly n sets she must win the n th set and exactly two of the first $n - 1$. In three sets: $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$. In four: $3 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) \times \frac{2}{3} = \frac{8}{27}$. In five: $6 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 \times \frac{2}{3} = \frac{16}{81}$. So $P(\text{Ana wins}) = \frac{24+24+16}{81} = \frac{64}{81}$, and the required probability is $\frac{24/81}{64/81} = \frac{3}{8}$. The option $\frac{8}{27}$ is the unconditional probability that Ana wins in four sets, $\frac{10}{27}$ is the probability of four sets with either winner, and $\frac{4}{5}$ is the reverse conditional, $P(\text{Ana wins} \mid \text{four sets})$.

Question 26. Answer **B**.

All $5! = 120$ allocations are equally likely. Let D_n be the number of ways in which none of n people gets their own present. Listing gives $D_2 = 1$, $D_3 = 2$ and $D_4 = 9$. For five people, friend 1 receives the present of one of the other four, say friend 2 (4 choices). Either friend 2 receives friend 1's present, and the other three are deranged in $D_3 = 2$ ways, or friend 2 does not, which is like a derangement of four in $D_4 = 9$ ways. So $D_5 = 4(2 + 9) = 44$. Exactly one person gets their own present in $5 \times D_4 = 45$ ways. The answer is $\frac{44 + 45}{120} = \frac{89}{120}$. The options $\frac{11}{30}$ and $\frac{3}{8}$ are the cases of nobody and exactly one separately.

Question 27. Answer **E**.



The figure shows the pyramid with the section used, and the section itself with the sphere appearing as its inscribed circle.

By symmetry the sphere's centre is on the axis and the sphere touches the base and all four triangular faces. The half-diagonal of the base is $3\sqrt{2}$, so the height is $\sqrt{36 - 18} = 3\sqrt{2}$, and the height of each triangular face, from the apex to the midpoint of a base edge, is $\sqrt{36 - 9} = 3\sqrt{3}$. Cut through the apex and the midpoints of two opposite base edges: the section is an isosceles triangle with base 6, equal sides $3\sqrt{3}$ and height $3\sqrt{2}$, and the sphere appears as its inscribed circle. So

$r = \frac{\text{area}}{\text{semi-perimeter}} = \frac{9\sqrt{2}}{3 + 3\sqrt{3}} = \frac{3\sqrt{2}}{1 + \sqrt{3}} = \frac{3\sqrt{6} - 3\sqrt{2}}{2}$. As a check, $r = \frac{3V}{S}$ with $V = 36\sqrt{2}$ and $S = 36 + 36\sqrt{3}$. The section through two opposite vertices (sides $6, 6, 6\sqrt{2}$) gives $6 - 3\sqrt{2}$, but that circle would poke out through the faces; $\frac{\sqrt{6}}{2}$ is the answer for a regular tetrahedron of edge 6.

