

HU Collection

ESAT Mock Set A: Mathematics 2

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	C	F	A	D	F	C	C	A	F	A

Q	11	12	13	14	15	16	17	18	19	20
Ans	A	B	C	A	F	A	B	F	C	A

Q	21	22	23	24	25	26	27			
Ans	E	B	D	E	E	F	E			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer C.

Since $(1+x+x^2)(1-x) = 1-x^3$, the expression is $(1-x^3)^4(1+x+x^2)^2 = (1-4x^3+6x^6-\dots)(1+2x+3x^2+2x^3+\dots)$. The second bracket has no x^5 term, so the only contribution is $(-4x^3)(3x^2)$, giving -12 . Taking the x^2 coefficient of $(1+x+x^2)^2$ to be 2 gives -8 , and forgetting the square altogether gives -4 .

Question 2. Answer F.

Do not multiply by $x^2 - 9$, whose sign varies. Subtracting $\frac{1}{2}$ gives $\frac{x^2 - 10x + 17}{2(x-3)(x+3)} \leq 0$. The numerator has roots $5 \pm 2\sqrt{2}$, roughly 2.17 and 7.83. For $-3 < x < 3$ the denominator is negative, so we need the numerator ≥ 0 : $-3 < x \leq 5 - 2\sqrt{2}$. For $x > 3$ we need the numerator ≤ 0 : $3 < x \leq 5 + 2\sqrt{2}$. For $x < -3$ both are positive. The integers are $-2, -1, 0, 1, 2$ and $4, 5, 6, 7$, nine in all. Multiplying through gives $5 - 2\sqrt{2} \leq x \leq 5 + 2\sqrt{2}$, which leads to 5 (or 4 after removing $x = 3$).

Question 3. Answer A.

By similarity the volume at depth h is $V = ah^3$ for a constant a . The constant rate is $\frac{dV}{dt} = 3ah^2 \frac{dh}{dt} = 3a(16)(9) = 432a$ per minute. The volume needed is $a(8^3 - 4^3) = 448a$, so the time is $\frac{448}{432} = \frac{28}{27}$ minutes. Assuming the level keeps rising at 9 cm per minute gives $\frac{4}{9}$.

Question 4. Answer D.

Distance: $5 + 2 \left(5 \cdot \frac{9}{16} + 5 \cdot \left(\frac{9}{16} \right)^2 + \dots \right) = 5 + 2 \cdot \frac{45/16}{7/16} = 5 + \frac{90}{7} = \frac{125}{7}$ m. Heights scale by $\frac{9}{16}$, so times scale by $\sqrt{9/16} = \frac{3}{4}$, and the time is $1 + 2 \left(\frac{3}{4} + \frac{9}{16} + \dots \right) = 1 + 2 \cdot \frac{3/4}{1/4} = 7$ s. The average speed is $\frac{125}{49} \text{ m s}^{-1}$. Using the ratio $\frac{9}{16}$ for the times as well gives a time of $\frac{25}{7}$ s and the tempting answer 5.

Question 5. Answer F.

I: $\log_2(8x) = 3 + \log_2 x$, a translation of 3 units upwards (it is also a stretch parallel to the x -axis with factor $\frac{1}{8}$). II: $y = 3 + \frac{1}{3} \log_2 x$. A stretch parallel to the y -axis keeps the point $(1, 0)$, but II passes through $(1, 3)$. A translation or a stretch parallel to the x -axis gives $y = \log_2(x - a) + b$, with gradient $\frac{1}{(x - a) \ln 2}$, which cannot equal the gradient $\frac{1}{3x \ln 2}$ of II for every x . So II needs two transformations. III: $\log_2(x^2) = 2 \log_2 |x|$ is defined for every $x \neq 0$ and has two branches, while any translation or stretch of $y = \log_2 x$ is defined only on a half-line. It agrees with the stretch $y = 2 \log_2 x$ only for $x > 0$. So only I works.

Question 6. Answer C.

Both bases must be positive and not equal to 1. As $x^2 - 5x + 7 = \left(x - \frac{5}{2}\right)^2 + \frac{3}{4} > 0$, this means $x > 1$, $x \neq 2$ and $x \neq 3$. Writing $L = \log_{x-1}(x^2 - 5x + 7)$, the second term is $\frac{2}{L}$, so $L^2 - 3L + 2 = 0$ and $L = 1$ or $L = 2$. $L = 1$ gives $x^2 - 5x + 7 = x - 1$, so $x = 2$ or $x = 4$, and only $x = 4$ is allowed (check: $\log_3 3 + 2 \log_3 3 = 3$). $L = 2$ gives $x^2 - 5x + 7 = (x - 1)^2$, so $x = 2$, which is not allowed. The only solution is $x = 4$. Forgetting that a base cannot be 1 gives 6 or 8.

Question 7. Answer C.

Rearranging, $x^2(1 - k) = 4 - k$. For $k = 1$ there is no solution. Otherwise $x^2 = \frac{4 - k}{1 - k}$, which is never equal to 1, so the excluded values $x = \pm 1$ never occur. A positive value of x^2 gives two solutions, zero gives one and a negative value gives none. The fraction is positive for $k < 1$ or $k > 4$, zero for $k = 4$ (only $x = 0$) and negative for $1 < k < 4$. So there are exactly two solutions when $k < 1$ or $k > 4$. The value $k = 4$ is the local maximum of the curve, touched at the single point $x = 0$.

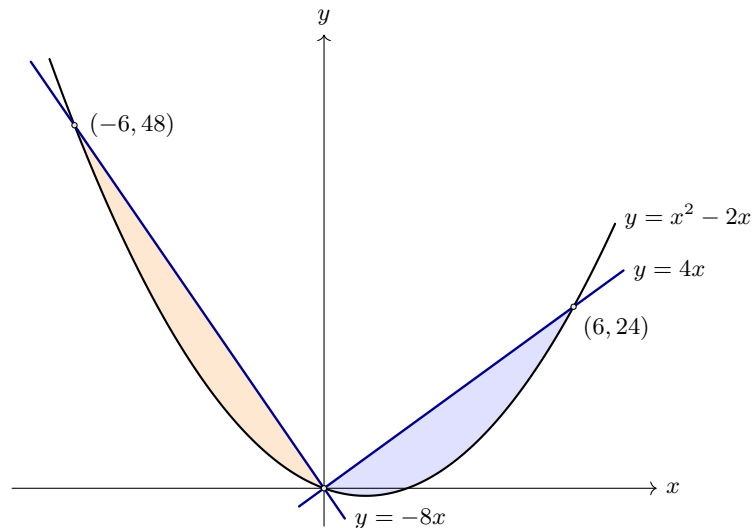
Question 8. Answer A.

f is increasing on its domain. If $f(x) = f^{-1}(x) = y$ then $f(y) = x$; if $x < y$ then $f(x) < f(y)$, that is $y < x$, a contradiction, and $x > y$ fails in the same way. So $y = x$ and $f(x) = x$, giving $x^2 - 7x + 10 = 0$, so $x = 2$ or $x = 5$, and only $x = 5$ lies in the domain. Solving $f(f(x)) = x$ instead also produces the roots of $x^2 - 5x + 5 = 0$; the root $\frac{5 + \sqrt{5}}{2} \approx 3.6$ is in the domain, but there $f(x) = \frac{5 - \sqrt{5}}{2}$, which is not $f^{-1}(x)$ because f^{-1} only takes values ≥ 3 .

Question 9. Answer F.

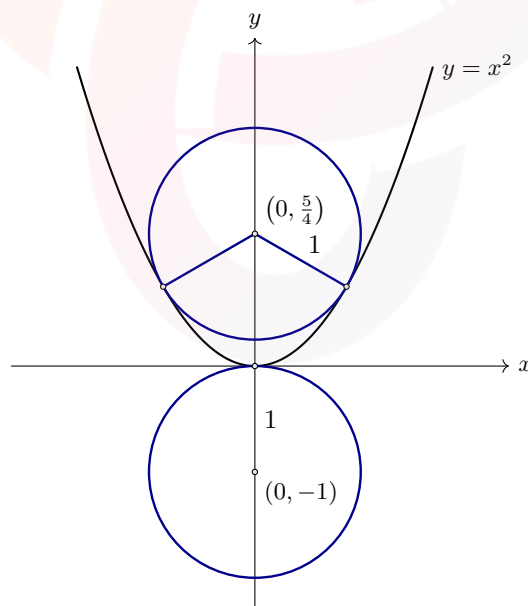
$\sin 5x = \sin x$ means $5x = x + 2k\pi$ or $5x = \pi - x + 2k\pi$. The first gives $x = \frac{k\pi}{2}$: five values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. The second gives $x = \frac{\pi}{6} + \frac{k\pi}{3}$: six values $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. The values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ appear in both lists, so there are $5 + 6 - 2 = 9$ solutions. Missing the overlap gives 11.

Question 10. Answer A.



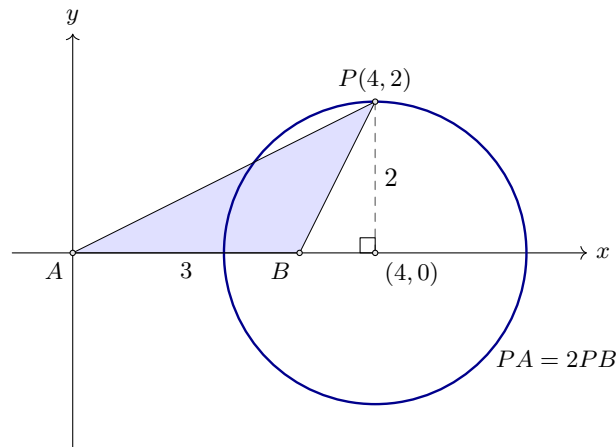
They meet where $x^2 - (m+2)x = 0$, at $x = 0$ and $x = m+2$. The area between $y = mx$ and the parabola is $\left| \int_0^{m+2} ((m+2)x - x^2) dx \right| = \frac{|m+2|^3}{6}$. Setting this equal to 36 gives $|m+2| = 6$, so $m = 4$ or $m = -8$, with sum -4 . The negative value is easy to miss, as is the shift by 2 (which gives $m = \pm 6$ and sum 0).

Question 11. Answer **A**.



The squared distance to (x, x^2) is $x^2 + (x^2 - c)^2 = u^2 + (1 - 2c)u + c^2$, where $u = x^2 \geq 0$. If $c \leq \frac{1}{2}$ this increases for $u \geq 0$, so the nearest point is the origin and the distance is $|c|$; with $c \leq \frac{1}{2}$ this gives only $c = -1$. If $c > \frac{1}{2}$ the minimum is at $u = c - \frac{1}{2}$ and equals $c^2 - \frac{(1 - 2c)^2}{4} = c - \frac{1}{4}$, so $c = \frac{5}{4}$. The sum is $\frac{1}{4}$. The tempting $c = 1$ fails: from $(0, 1)$ the nearest points are $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, at distance $\frac{\sqrt{3}}{2}$.

Question 12. Answer **B**.



With $P(x, y)$, $x^2 + y^2 = 4((x - 3)^2 + y^2)$ simplifies to $x^2 + y^2 - 8x + 12 = 0$, that is $(x - 4)^2 + y^2 = 4$, a circle of radius 2. The base $AB = 3$ is fixed, and the height $|y|$ is at most 2, reached at $(4, \pm 2)$. The greatest area is $\frac{1}{2} \times 3 \times 2 = 3$.

Question 13. Answer **C**.

Compare each with simple fractions. Since $3^2 > 2^3$, $a > \frac{3}{2}$; since $5^2 < 3^3$, $b < \frac{3}{2}$. So $a > b$. Next, $5^3 = 125 > 81 = 3^4$, so $b > \frac{4}{3}$, while $8^3 = 512 < 625 = 5^4$, so $c < \frac{4}{3}$. Hence $a > b > c$.

Question 14. Answer **A**.

The sum is $\log_2$ of the product $\frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{3} \cdots \frac{n+2}{n}$. The numerators are $3, 4, \dots, n+2$ and the denominators are $1, 2, \dots, n$, so everything cancels except $\frac{(n+1)(n+2)}{1 \times 2}$. We need $(n+1)(n+2) > 2^{11} = 2048$. Since $44 \times 45 = 1980$ and $45 \times 46 = 2070$, the answer is $n = 44$. Cancelling as if adjacent terms telescoped leaves $n+2$ and gives 1023; forgetting the 2 gives 31.

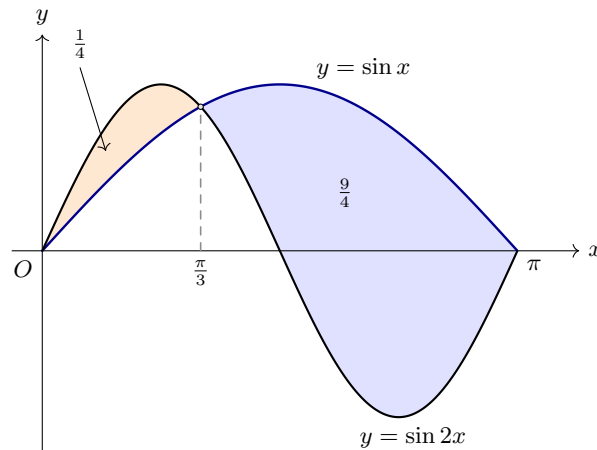
Question 15. Answer **F**.

$\frac{dy}{dx} = 4x^3 - 12x^2 + 2ax = 2x(2x^2 - 6x + a)$, so $x = 0$ is always a stationary point. There must be no other, so $2x^2 - 6x + a = 0$ must have no root other than 0. If $36 - 8a < 0$, that is $a > \frac{9}{2}$, it has no roots. If $a = \frac{9}{2}$ it has the repeated root $x = \frac{3}{2}$, which is a stationary point of inflexion, still a stationary point, so there are two. If $a = 0$ the roots are 0 and 3, again two stationary points. So $a > \frac{9}{2}$.

Question 16. Answer **A**.

Write $3 \sin x + 4 \cos x = 5 \sin(x + \alpha)$, where α is acute with $\sin \alpha = \frac{4}{5}$. As x runs from 0 to $\frac{3\pi}{2}$, $x + \alpha$ runs from α to $\frac{3\pi}{2} + \alpha$, so the value starts at 4, rises to 5, falls to -5 (when $x + \alpha = \frac{3\pi}{2}$), then rises to -3 at $x = \frac{3\pi}{2}$. Counting: $n = 5$ once; $n = 4$ twice ($x = 0$ and once on the way down); each of $3, 2, \dots, -4$ once on the way down, 8 pairs; $n = -5$ once; then -4 and -3 again on the way up. The total is $1 + 2 + 8 + 1 + 2 = 14$. Counting each attainable integer once gives 11.

Question 17. Answer **B**.



The curves meet where $\sin x(2 \cos x - 1) = 0$, at $x = 0, \frac{\pi}{3}, \pi$. On $\left[0, \frac{\pi}{3}\right]$, $\sin 2x$ is on top:

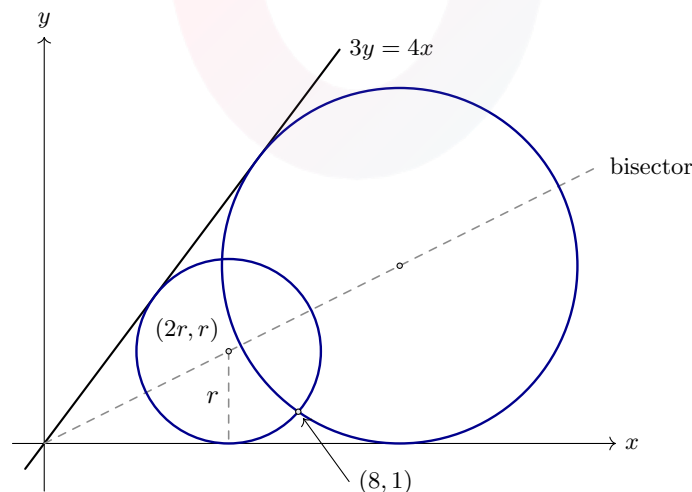
$$\left[-\frac{1}{2} \cos 2x + \cos x\right]_0^{\pi/3} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}. \text{ On } \left[\frac{\pi}{3}, \pi\right], \sin x \text{ is on top: } \left[-\cos x + \frac{1}{2} \cos 2x\right]_{\pi/3}^{\pi} = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}.$$

The total is $\frac{5}{4}$. Integrating $\sin x - \sin 2x$ over the whole interval gives 2, because the small region is counted as negative.

Question 18. Answer **F**.

If the roots are r and s , then $r + s = k$ and $rs = k + 3$, so $rs - r - s = 3$, that is $(r - 1)(s - 1) = 4$. The factorisations 1×4 , 2×2 , $(-1) \times (-4)$, $(-2) \times (-2)$ give root pairs $\{2, 5\}$, $\{3, 3\}$, $\{0, -3\}$, $\{-1, -1\}$, and so $k = 7, 6, -3, -2$. The sum is 8. Forgetting negative factors gives 13; forgetting repeated roots gives 4.

Question 19. Answer **C**.



The point $(8, 1)$ lies inside the angle between the positive x -axis and the ray along $(3, 4)$. A circle touching both lines cannot cross either of them, so it lies inside this angle and its centre is on the angle bisector. Adding unit vectors, $(1, 0) + \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{8}{5}, \frac{4}{5}\right)$, so the bisector has direction $(2, 1)$

and a circle of radius r has centre $(2r, r)$ (check: its distance from $4x - 3y = 0$ is $\frac{|8r - 3r|}{5} = r$). Passing through $(8, 1)$ gives $(8 - 2r)^2 + (1 - r)^2 = r^2$, that is $4r^2 - 34r + 65 = 0$. The discriminant

$34^2 - 16 \times 65 = 116$ is positive and both roots are positive, so the sum of the radii is $\frac{34}{4} = \frac{17}{2}$. Taking the bisector to have the average gradient $\frac{2}{3}$ gives $\frac{104}{9}$.

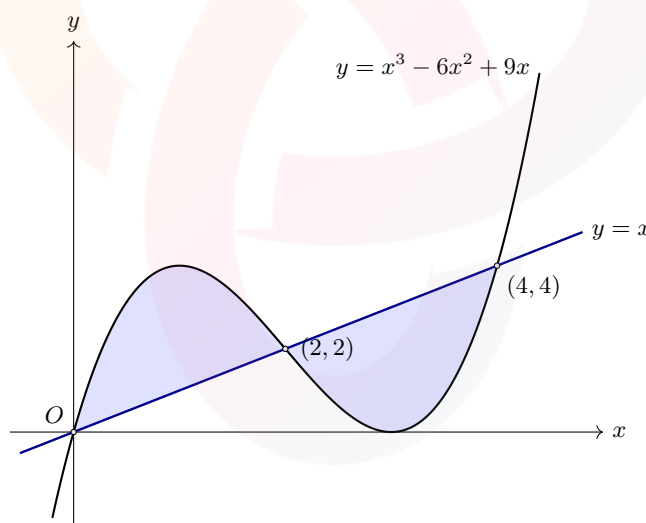
Question 20. Answer **A**.

Let $y = \sqrt{x+6} \geq 0$. Then $y^2 = x+6$ and $y = x^2 - 6$; subtracting, $y^2 - x^2 = x - y$, so $(y-x)(y+x+1) = 0$. If $y = x$, then $x^2 - x - 6 = 0$, so $x = 3$ or $x = -2$, and $y = x \geq 0$ leaves $x = 3$. If $y = -1 - x$, then $x^2 + x - 5 = 0$, so $x = \frac{-1 \pm \sqrt{21}}{2}$, and $y \geq 0$ needs $x \leq -1$, leaving $x = \frac{-1 - \sqrt{21}}{2}$. The sum is $\frac{5 - \sqrt{21}}{2}$. Squaring gives the quartic $x^4 - 12x^2 - x + 30 = 0$; all four of its roots are real and add to 0, but two of them are extraneous.

Question 21. Answer **E**.

Let $u = f(x)$. Then $u^3 - 3u - 2 = (u+1)^2(u-2) = 0$, so $u = -1$ or $u = 2$. The graph of f has a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$. Since $-2 < -1 < 2$, the line $y = -1$ meets the graph three times. The line $y = 2$ touches the graph at the maximum $x = -1$ and crosses it once more at $x = 2$, giving only two distinct solutions. The total is $3 + 2 = 5$. Assuming every horizontal line between turning values gives three solutions leads to 6, and counting the degree gives 9.

Question 22. Answer **B**.



No integration is needed. Every cubic is symmetric under a half turn about its point of inflection. Here $y'' = 6x - 12 = 0$ at $x = 2$, where $y = 8 - 24 + 18 = 2$, so the half turn about $(2, 2)$ maps the curve to itself. It also maps any line through $(2, 2)$ to itself, so such a line cuts off two congruent regions, one on each side of $(2, 2)$. The line through the origin and $(2, 2)$ is $y = x$, so $m = 1$. Check: $x^3 - 6x^2 + 9x = x$ gives $x(x-2)(x-4) = 0$, three distinct points placed symmetrically about $x = 2$. It is the only such line: for $0 < m < 9$ the other intersections are $x = 3 \pm \sqrt{m}$, and making the signed area from 0 to $b = 3 + \sqrt{m}$ vanish gives $b^2 - 8b + 18 - 2m = 0$, which with $m = (b-3)^2$ becomes $b(b-4) = 0$, so $b = 4$ and $m = 1$; for $m > 9$ the region to the left of the origin is always much the smaller. Working out one area and halving, or using the tangent gradient at the origin (9), leads to the wrong options.

Question 23. Answer **D**.

Expanding $\sin(2x \pm x)$ and $\cos(2x \pm x)$ gives $\sin x + \sin 3x = 2 \sin 2x \cos x$ and $\cos x + \cos 3x = 2 \cos 2x \cos x$. So the equation is $(1 + 2 \cos x)(\sin 2x - \cos 2x) = 0$. Either $\cos x = -\frac{1}{2}$, giving $x = \frac{2\pi}{3}$,

or $\tan 2x = 1$ with $0 \leq 2x \leq 2\pi$, giving $x = \frac{\pi}{8}, \frac{5\pi}{8}$. The sum is $\frac{17\pi}{12}$. Cancelling the common factor $1 + 2\cos x$ loses $\frac{2\pi}{3}$ and gives $\frac{3\pi}{4}$.

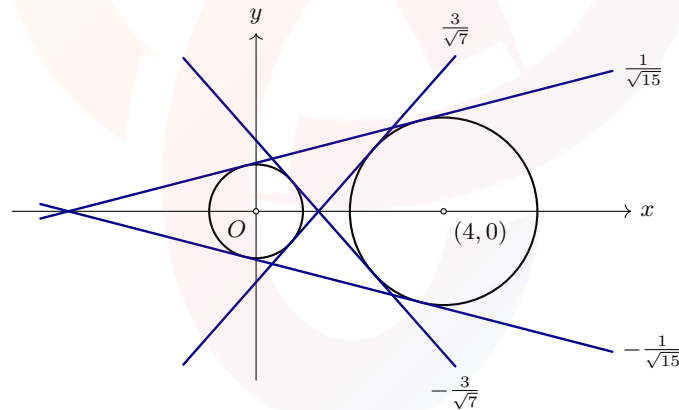
Question 24. Answer **E**.

Integrate the binomial expansion $(1+x)^{10} = \sum_{k=0}^{10} \binom{10}{k} x^k$ from 0 to 1. Each term gives $\frac{1}{k+1} \binom{10}{k}$, so the sum is $\int_0^1 (1+x)^{10} dx = \left[\frac{(1+x)^{11}}{11} \right]_0^1 = \frac{2^{11} - 1}{11} = \frac{2047}{11}$. Forgetting the lower limit gives $\frac{2048}{11}$.

Question 25. Answer **E**.

Note $x = 0$ is never a solution. For $x < 0$, $g(x) = e^x - kx^2$ has $g'(x) = e^x - 2kx > 0$, with $g(0) = 1 > 0$ and $g(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, so there is exactly one negative solution for every $k > 0$. For $x > 0$, the equation is $h(x) = \frac{e^x}{x^2} = k$, and $h'(x) = \frac{e^x(x-2)}{x^3}$. So h falls from $+\infty$ to its minimum $h(2) = \frac{e^2}{4}$ and then rises to $+\infty$, giving two positive solutions when $k > \frac{e^2}{4}$ and one when $k = \frac{e^2}{4}$. Exactly three solutions occur for $k > \frac{e^2}{4}$. The tempting $k \geq \frac{e^2}{4}$ fails because at $k = \frac{e^2}{4}$ the curves only touch at $x = 2$, giving two solutions in total.

Question 26. Answer **F**.



No vertical line touches both circles, so each common tangent is $y = mx + c$. Tangency to the first circle gives $c^2 = 1 + m^2$, and to the second gives $(4m + c)^2 = 4(1 + m^2) = 4c^2$. So $4m + c = \pm 2c$, that is $c = 4m$ or $c = -\frac{4m}{3}$. Then $16m^2 = 1 + m^2$ gives $m^2 = \frac{1}{15}$, and $\frac{16m^2}{9} = 1 + m^2$ gives $m^2 = \frac{9}{7}$. The gradients are $\pm \frac{1}{\sqrt{15}}$ and $\pm \frac{3}{\sqrt{7}}$, with product $\left(-\frac{1}{15}\right) \left(-\frac{9}{7}\right) = \frac{3}{35}$.

Question 27. Answer **E**.

The tangent at $x = t$ is $y = (3t^2 - 1)(x - t) + t^3 - t = (3t^2 - 1)x - 2t^3$. It passes through $(2, k)$ when $p(t) = 2t^3 - 6t^2 + 2 + k = 0$. A line cannot touch a cubic at two different points, so distinct values of t give distinct tangents, and we need three distinct real roots. As $p'(t) = 6t(t - 2)$, this happens when the local maximum $p(0) = 2 + k$ is positive and the local minimum $p(2) = k - 6$ is negative, so $-2 < k < 6$. That is 7 integers. At $k = -2$ and $k = 6$ (the point $(2, 6)$ is on the curve) there are only two tangents.