

HU Collection

ESAT Mock Set A: Mathematics 1

Worked Solutions

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Answer key

Q	1	2	3	4	5	6	7	8	9	10
Ans	C	F	D	D	C	A	B	F	C	B

Q	11	12	13	14	15	16	17	18	19	20
Ans	E	F	D	B	E	F	A	F	B	B

Q	21	22	23	24	25	26	27			
Ans	C	F	C	A	F	A	D			

Each solution gives an intended route and names the trap the question was built around. Diagrams show the configuration each solution works with.

Question 1. Answer C.

The base has area 3 m^2 , so each centimetre of depth is $0.03 \text{ m}^3 = 30 \text{ litres}$.

For the first 10 minutes tap A alone cannot beat the drain ($15 < 18$), so the tank stays empty: the water level cannot go negative. From then on the net rate is $15 + 9 - 18 = 6 \text{ litres per minute}$, so reaching 12 cm (360 litres) takes 60 minutes. With the drain closed the rate is 24 litres per minute, and the remaining 48 cm (1440 litres) takes another 60 minutes. The total is $10 + 60 + 60 = 130$ minutes.

The trap is to let the tank go to -30 litres in the first ten minutes, which adds 5 minutes and gives 2 hours 15 minutes.

Question 2. Answer F.

In a histogram, frequency is proportional to area, not height. The $10 \leq t < 15$ bar has area $5 \times 3 = 15$ units for 30 runners, so each unit of area is 2 runners. The frequencies are $10 \times 1 \times 2 = 20$, 30 , $5 \times 4 \times 2 = 40$ and $20 \times 0.75 \times 2 = 30$, so $n = 120$.

The cumulative frequencies at $t = 10, 15, 20, 40$ are 20, 50, 90, 120. The lower quartile (the 30th value) is 10 runners into the second class: $10 + 5 \times \frac{10}{30} = \frac{35}{3}$. The upper quartile (the 90th value) is exactly at $t = 20$. So the interquartile range is $20 - \frac{35}{3} = \frac{25}{3}$.

Taking the bar heights as proportional to frequency, or reading off the lower quartile at a class boundary, gives the popular wrong answers.

Question 3. Answer D.

If $k = 1$ the equation is linear, $-4x + 3 = 0$, which has exactly one solution. Otherwise it is a quadratic, with exactly one real solution when the discriminant is zero:

$$16 - 4(k-1)(k+2) = 0 \iff k^2 + k - 6 = 0 \iff k = -3 \text{ or } k = 2.$$

So there are 3 values. The usual answer is 2, from forgetting that the x^2 term can vanish.

Question 4. Answer D.

Put $u = x^2 - 3x$. Then $u^2 + u - 6 = 0$, so $u = 2$ or $u = -3$.

The equation $x^2 - 3x = 2$ has discriminant $9 + 8 = 17 > 0$, so two real roots with sum 3. The equation $x^2 - 3x + 3 = 0$ has discriminant $9 - 12 < 0$, so no real roots. (Equivalently, $x^2 - 3x = (x - \frac{3}{2})^2 - \frac{9}{4} \geq -\frac{9}{4}$, so $u = -3$ is impossible.)

The sum is 3. Adding the root sums of both quadratics without checking gives 6.

Question 5. Answer **C**.

Exactly three months means one pair shares a month and the other two people have different months from each other and from the pair. Choose the pair in $\binom{4}{2} = 6$ ways, then the months for the pair and the two others in $12 \times 11 \times 10$ ways.

$$P = \frac{6 \times 12 \times 11 \times 10}{12^4} = \frac{7920}{20736} = \frac{55}{144}.$$

Leaving out the factor 6 gives $\frac{55}{864}$. The option $\frac{41}{96}$ is the probability that at least two share a month, which also includes three or four sharing and two separate pairs.

Question 6. Answer **A**.

Since x and y both appear in the numerator and the denominator, you cannot simply take the largest numerator over the smallest denominator: that would need x at its upper bound and at its lower bound at the same time. Instead write

$$\frac{x+y}{x-y} = 1 + \frac{2y}{x-y},$$

which increases as y increases and as x decreases. So take $x = 11.5$ and $y = 4.5$, giving $\frac{16}{7}$. The tempting $\frac{17}{7}$ comes from $\frac{12.5 + 4.5}{11.5 - 4.5}$, which uses two different values of x .

Question 7. Answer **B**.

Use the difference of two squares: $2^{20} - 1 = (2^{10} - 1)(2^{10} + 1) = 1023 \times 1025$. Now $1023 = 3 \times 11 \times 31$ and $1025 = 25 \times 41$, so

$$2^{20} - 1 = 3 \times 5^2 \times 11 \times 31 \times 41.$$

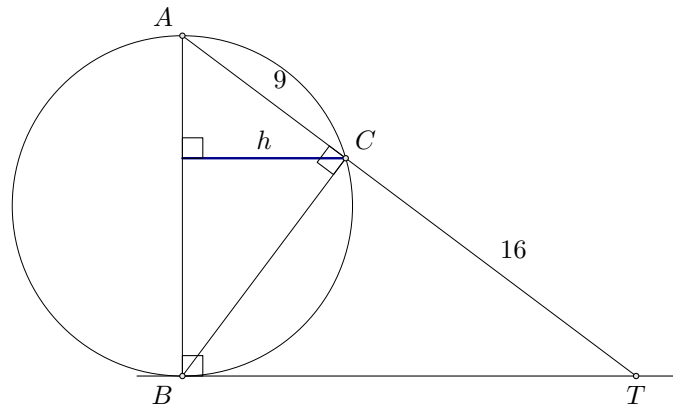
So 3, 5, 11, 25, 31, 41 are factors and 7 and 13 are not: 6 in total. Splitting further as $(2^5 - 1)(2^5 + 1)(2^{10} + 1)$ gives the same factors; it is easy to forget that 25, not just 5, divides 1025.

Question 8. Answer **F**.

Let the speeds be p and q . After they pass, Priya still has to cover the stretch Quentin rode in the first 2 hours, namely $2q$, and she does it in $\frac{4}{3}$ hours: $2q = \frac{4}{3}p$, so $q = \frac{2}{3}p$.

Quentin must similarly cover Priya's first stretch $2p$ at speed q , which takes $\frac{2p}{q} = 3$ hours. His whole journey is $2 + 3 = 5$ hours. (In general the meeting time is the geometric mean of the two times after meeting: $2^2 = \frac{4}{3} \times 3$.) The option 3 hours stops at the time after passing; 3 hours 20 minutes is Priya's journey.

Question 9. Answer **C**.



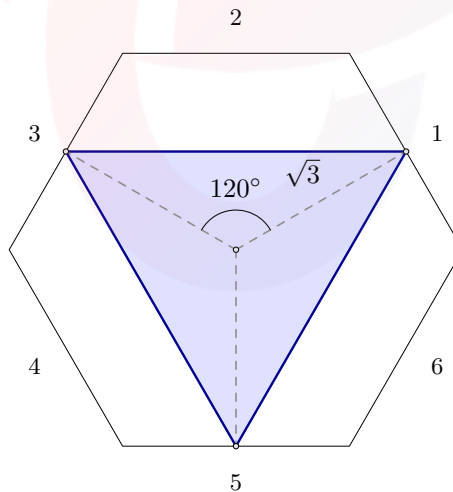
The tangent is perpendicular to the diameter, so $\angle ABT = 90^\circ$, and the angle in a semicircle gives $\angle ACB = 90^\circ$. So BC is the altitude from the right angle of triangle ABT , whose hypotenuse is $AT = 25$.

Triangles ACB and ABT are similar (common angle A , each with a right angle), so $AB^2 = AC \times AT = 225$ and $AB = 15$; then $BC = \sqrt{225 - 81} = 12$. In the right-angled triangle ACB , the distance from C to AB is the altitude to the hypotenuse:

$$h = \frac{AC \times BC}{AB} = \frac{9 \times 12}{15} = \frac{36}{5}.$$

The options $\frac{27}{5}$ and $\frac{48}{5}$ are the two pieces into which this altitude cuts AB .

Question 10. Answer **B**.



Take the hexagon with side 2, made of six equilateral triangles of side 2, so its area is $6\sqrt{3}$. The midpoint of each side is at distance $\sqrt{3}$ from the centre (the height of one of those equilateral triangles), and the three chosen midpoints are 120° apart.

So the triangle is equilateral with circumradius $\sqrt{3}$, hence side $\sqrt{3} \times \sqrt{3} = 3$ and area $\frac{\sqrt{3}}{4} \times 9 = \frac{9\sqrt{3}}{4}$. The ratio to the hexagon is $\frac{9}{24} = \frac{3}{8}$, and $\frac{3}{8} \times 60 = 22.5$. The option 30 is the triangle on alternate vertices, which is half the hexagon.

Question 11. Answer **E**.

Any prime other than 2 or 3 in n would only make n bigger, so let $n = 2^a 3^b$. Then $12n = 2^{a+2} 3^{b+1}$ is a square when a is even and b is odd, and $18n = 2^{a+1} 3^{b+2}$ is a cube when $a \equiv 2$ and $b \equiv 1 \pmod{3}$.

The smallest choices are $a = 2$ and $b = 1$, so $n = 12$. Check: $12 \times 12 = 144 = 12^2$ and $18 \times 12 = 216 = 6^3$.

Question 12. Answer **F**.

The general term is $\frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}$. Take out $\sqrt{k}\sqrt{k+1}$ and rationalise:

$$\frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1} + \sqrt{k})} = \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k}\sqrt{k+1}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}.$$

Summing from $k = 1$ to 99 telescopes to $1 - \frac{1}{\sqrt{100}} = \frac{9}{10}$. Writing $\frac{1}{k} - \frac{1}{k+1}$ by mistake gives $\frac{99}{100}$.

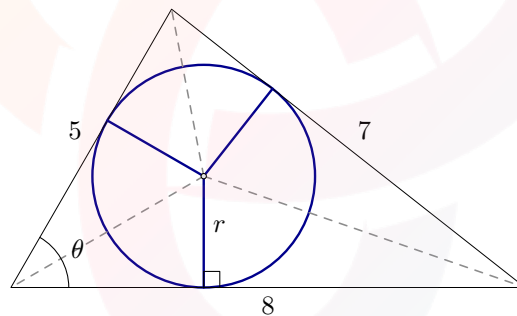
Question 13. Answer **D**.

Factorise: $x^6 - \frac{1}{x^6} = \left(x^3 - \frac{1}{x^3}\right)\left(x^3 + \frac{1}{x^3}\right)$.

With $t = x - \frac{1}{x} = 3$, cubing gives $x^3 - \frac{1}{x^3} = t^3 + 3t = 36$. Next, $\left(x + \frac{1}{x}\right)^2 = t^2 + 4 = 13$, and since $x > 0$, $s = x + \frac{1}{x} = \sqrt{13}$. Then $x^3 + \frac{1}{x^3} = s^3 - 3s = 13\sqrt{13} - 3\sqrt{13} = 10\sqrt{13}$.

So the answer is $36 \times 10\sqrt{13} = 360\sqrt{13}$. Ignoring the condition $x > 0$ allows the negative value; using $x^3 + \frac{1}{x^3} = 36$ as well gives 1296.

Question 14. Answer **B**.



By the cosine rule, the angle θ opposite the side 7 has $\cos \theta = \frac{25 + 64 - 49}{2 \times 5 \times 8} = \frac{1}{2}$, so $\theta = 60^\circ$ and the area is $\frac{1}{2} \times 5 \times 8 \times \sin 60^\circ = 10\sqrt{3}$.

Splitting the triangle into three triangles from the incentre gives area $= r \times s$, where $s = 10$ is the semi-perimeter. So $r = \frac{10\sqrt{3}}{10} = \sqrt{3}$. Using the full perimeter gives $\frac{\sqrt{3}}{2}$; the option $\frac{7\sqrt{3}}{3}$ is the circumradius.

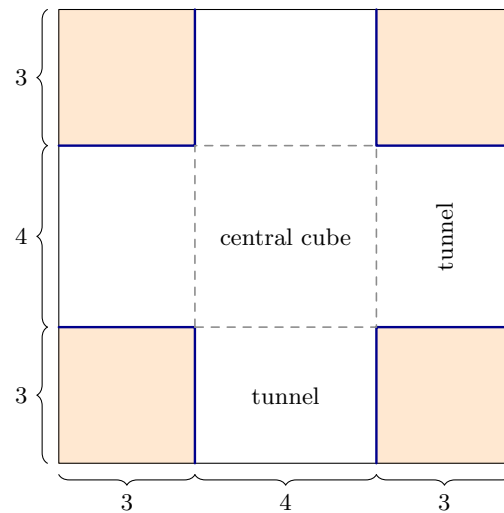
Question 15. Answer **E**.

The mean is $\frac{37+x}{5}$. The median depends on where x sits.

If $x \leq 7$ the median is 7, so $37 + x = 35$ and $x = -2$, which is valid. If $7 \leq x \leq 12$ the median is x , so $37 + x = 5x$ and $x = \frac{37}{4}$, valid. If $x \geq 12$ the median is 12, so $x = 23$, valid.

The sum is $-2 + \frac{37}{4} + 23 = \frac{121}{4}$. Assuming x must be positive (it need not be) loses -2 and gives $\frac{129}{4}$.

Question 16. Answer **F**.



The figure shows the cross-section through the centre of the cube, parallel to one face: the shaded corner squares are wood and the white cross is the tunnels.

Outside: each face loses a 4×4 square, leaving $6 \times (100 - 16) = 504$.

Inside: the three tunnels share a central $4 \times 4 \times 4$ cube, all of whose faces are open, so there are no walls there. From each face, a tunnel arm runs $\frac{10-4}{2} = 3$ cm in to the central cube, and has four walls of 4×3 , so area 48. There are 6 arms, giving 288.

The total is $504 + 288 = 792$. Treating each tunnel as having full length walls ($3 \times 4 \times 40 = 480$) gives 984, which counts walls through the hollow centre.

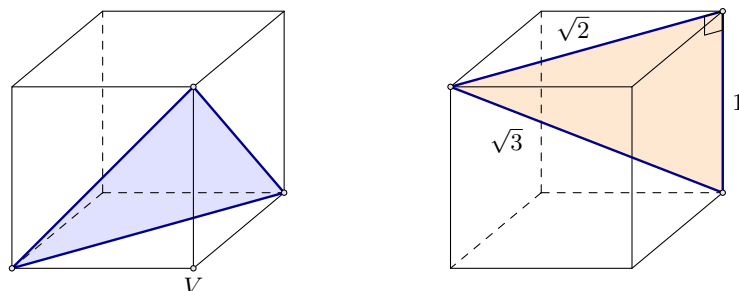
Question 17. Answer A.

The second blue comes before the third red exactly when at least two of the first four counters are blue (in the first four draws, one of these two events must happen, and they cannot both).

Every set of four counters is equally likely to be the first four, out of $\binom{7}{4} = 35$. All four red: 1 way. Exactly one blue: $3 \times \binom{4}{3} = 12$ ways. So the probability is $1 - \frac{13}{35} = \frac{22}{35}$.

The option $\frac{13}{35}$ is the complement, and $\frac{18}{35}$ counts only exactly two blues among the first four.

Question 18. Answer F.

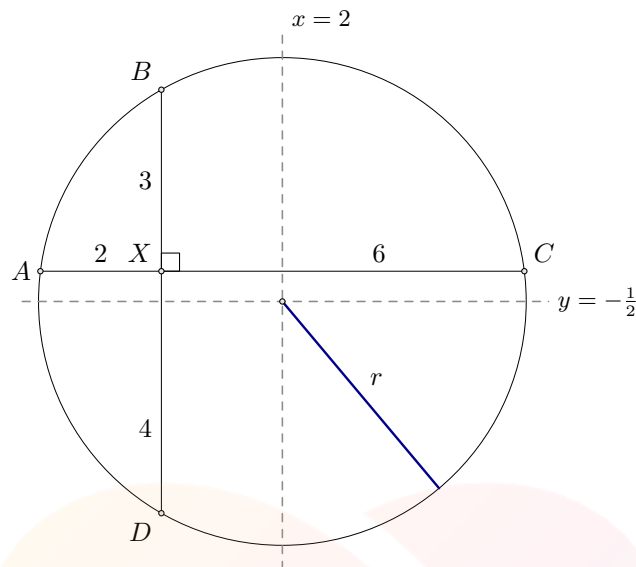


There are $\binom{8}{3} = 56$ sets, and no three vertices of a cube are collinear. It is easier to count the triangles that are *not* right-angled.

Take a vertex V and its three neighbours: the neighbours are joined to each other by face diagonals, so they form an equilateral triangle. This gives 8 equilateral triangles, one for each vertex. Every other triangle contains an edge of the cube, and its sides are either $1, 1, \sqrt{2}$ (half a face) or $1, \sqrt{2}, \sqrt{3}$ (an edge, a face diagonal and a space diagonal), taking the edge as 1. Both satisfy Pythagoras, so all of these are right-angled. So the probability is $\frac{56-8}{56} = \frac{6}{7}$.

Counting only the triangles lying in a face, $6 \times 4 = 24$, gives the trap answer $\frac{3}{7}$.

Question 19. Answer **B**.



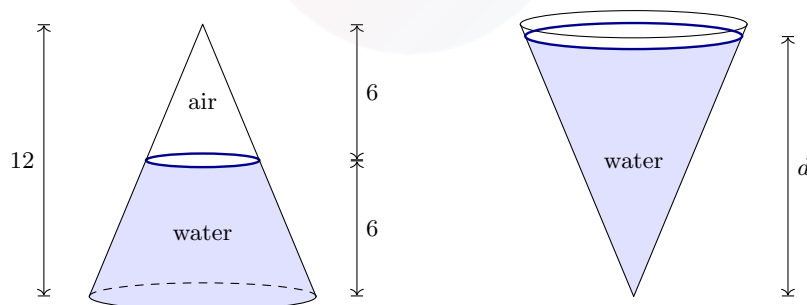
Triangles AXB and DXC are similar (angles in the same segment), so $AX \times XC = BX \times XD$, giving $XD = \frac{12}{3} = 4$.

Now use coordinates with X at the origin: $A = (-2, 0)$, $C = (6, 0)$, $B = (0, 3)$, $D = (0, -4)$. The centre lies on the perpendicular bisector of AC , the line $x = 2$, and on that of BD , the line $y = -\frac{1}{2}$. So the centre is $(2, -\frac{1}{2})$ and

$$r^2 = (6 - 2)^2 + \left(\frac{1}{2}\right)^2 = \frac{65}{4}, \quad r = \frac{\sqrt{65}}{2}.$$

This is just over 4, which is the tempting answer if you assume AC is a diameter.

Question 20. Answer **B**.



With the vertex up, the air fills a small cone of height 6, which is similar to the whole cone with scale factor $\frac{1}{2}$, so it has $\frac{1}{8}$ of the volume. The water is therefore $\frac{7}{8}$ of the cone.

With the vertex down, the water forms a cone of depth d similar to the whole cone, so $\left(\frac{d}{12}\right)^3 = \frac{7}{8}$

and $d = 12 \times \frac{\sqrt[3]{7}}{2} = 6\sqrt[3]{7} \approx 11.5$. Thinking the water fills half the volume gives $6\sqrt[3]{4}$; using the scale factor linearly gives 10.5.

Question 21. Answer **C**.

Let the escalator have N visible steps and move at e steps per second. At 1 step per second Ria walks for 30 seconds, while the escalator carries her $30e$ steps, so $N = 30 + 30e$. At 3 steps per second she walks for 15 seconds, so $N = 45 + 15e$.

Equating, $15e = 15$, so $e = 1$ and $N = 60$. The common error is to treat the two trips as taking the same time.

Question 22. Answer **F**.

Rearrange and factorise: $xy - 4x - 6y + 24 = 24$, so

$$(x - 6)(y - 4) = 24.$$

Every way of writing 24 as a product of two integers $d \times \frac{24}{d}$ gives exactly one pair, with $x = 6 + d$. The number 24 has 8 positive divisors, and d may also be negative, so there are 16 pairs (including $(0, 0)$). Counting only positive divisors gives 8.

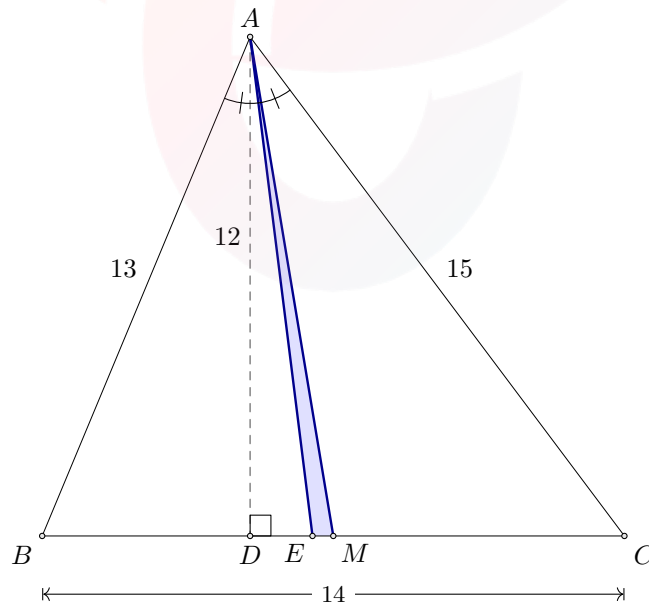
Question 23. Answer **C**.

Let p be the probability that Asha wins. A full round with no winner happens with probability $\frac{5}{6} \times \frac{4}{6} = \frac{5}{9}$, after which the game is in its starting position again. So

$$p = \frac{1}{6} + \frac{5}{9}p \implies \frac{4}{9}p = \frac{1}{6} \implies p = \frac{3}{8}.$$

Going first does not make up for Bilal's doubled chance on each turn. The option $\frac{6}{11}$ is the answer when both need a 6; $\frac{1}{3}$ is Asha's share of the single roll chances $\frac{1}{6} : \frac{2}{6}$, which ignores the turn order.

Question 24. Answer **A**.



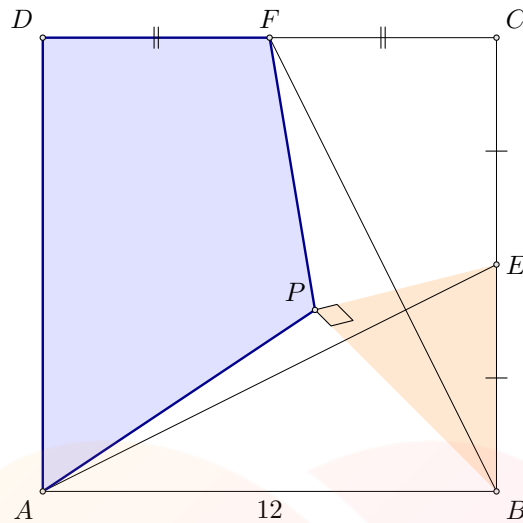
Drop the perpendicular AD to BC and let $BD = p$. Then $13^2 - p^2 = 15^2 - (14 - p)^2$, giving $p = 5$ and $AD = 12$.

The angle bisector theorem gives $BE : EC = AB : AC = 13 : 15$, so $BE = \frac{13}{28} \times 14 = 6.5$, while $BM = 7$. Hence $EM = 0.5$, and triangle AEM has base EM on BC and height AD :

$$\text{area} = \frac{1}{2} \times 0.5 \times 12 = 3.$$

The option 9 is the area of triangle ADE (with $DE = 1.5$), and 12 is the area of triangle ADM (with $DM = 2$).

Question 25. Answer **F**.



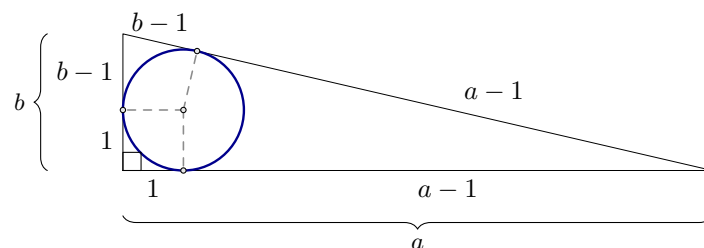
$APFD$ is what is left of the square after removing the union of triangles ABE and BCF . Each has area $\frac{1}{2} \times 12 \times 6 = 36$, and they overlap in triangle BPE .

Triangles ABE and BCF are congruent, and rotating one onto the other by 90° shows $AE \perp BF$. So BPE is right-angled at P and similar to ABE (which is right-angled at B), with scale factor $\frac{BE}{AE} = \frac{6}{6\sqrt{5}} = \frac{1}{\sqrt{5}}$. Its area is $\frac{36}{5}$.

$$\text{area of } APFD = 144 - \left(36 + 36 - \frac{36}{5} \right) = 72 + \frac{36}{5} = \frac{396}{5}.$$

Forgetting the overlap gives 72; adding it twice gives $\frac{432}{5}$.

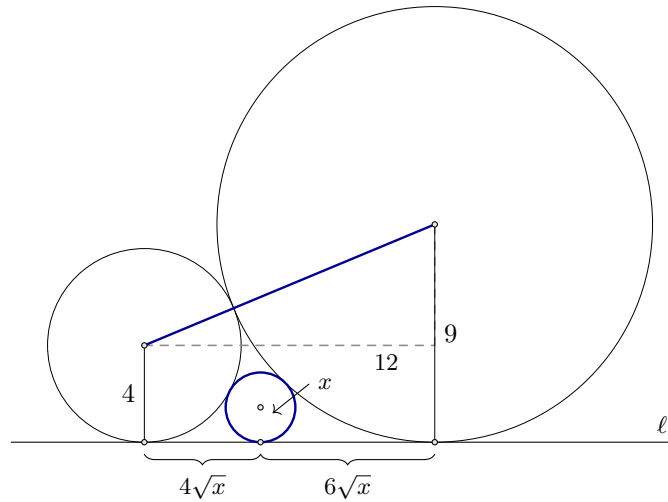
Question 26. Answer **A**.



Let the legs be a and b . The incircle touches the two legs at distance $r = 1$ from the right-angled vertex (the centre and the two touching points form a square), and the two tangents from each other vertex are equal. So the hypotenuse is $(a - 1) + (b - 1)$, giving $a + b = 12$.

Then $2ab = (a + b)^2 - (a^2 + b^2) = 144 - 100 = 44$, so the area is $\frac{1}{2}ab = 11$. (The legs are $6 \pm \sqrt{14}$, so such a triangle exists.) Guessing the familiar 6, 8, 10 triangle gives 24, but that triangle has inradius 2.

Question 27. Answer **D**.



First find the distance between the points where two touching circles of radii R and r meet a common line. Their centres are $R + r$ apart and at heights R and r , so by Pythagoras the horizontal distance is $\sqrt{(R + r)^2 - (R - r)^2} = 2\sqrt{Rr}$.

The touching points of the circles of radii 4 and 9 are $2\sqrt{36} = 12$ apart. If the small circle has radius x , its touching point lies between them, $2\sqrt{4x} = 4\sqrt{x}$ from the first and $2\sqrt{9x} = 6\sqrt{x}$ from the second. So

$$4\sqrt{x} + 6\sqrt{x} = 12 \implies \sqrt{x} = \frac{6}{5} \implies x = \frac{36}{25}.$$

Stopping at $\sqrt{x}$ gives $\frac{6}{5}$; the option $\frac{36}{13} = \frac{4 \times 9}{4 + 9}$ is a guess at a harmonic mean.