

British Mathematical Olympiad Round 2 2011

Solutions and Insights

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July 25, 2026

Overview

The British Mathematical Olympiad Round 2 (BMO2) is a highly challenging 3.5-hour examination consisting of four problems, serving as a critical selection tool for the UK team for the International Mathematical Olympiad (IMO). The 2011 paper features a beautiful mix of synthetic geometry, number theory, functional equations/sequences, and extremal combinatorics. The problems demand not only deep mathematical insight but also rigorous, clear, and elegant proof writing.

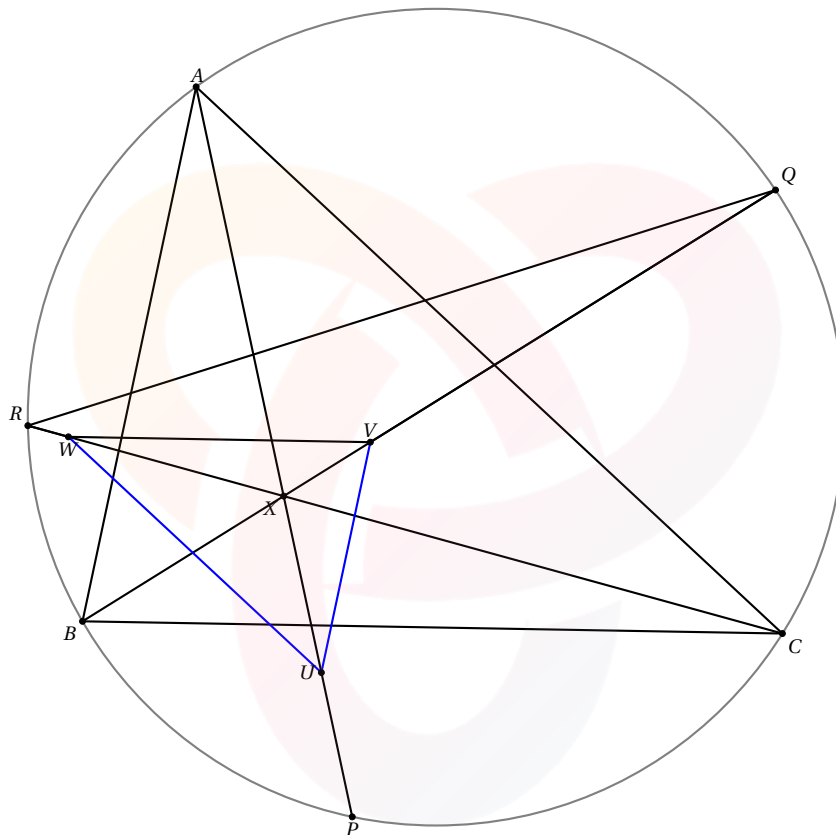
Remark

Author's Note: The solutions presented for these four problems are original write-ups and have not been copied from official mark schemes, which aren't published online. Consequently, they may contain minor algebraic errors, typos, or oversights. Readers are warmly encouraged to review the arguments critically and submit any corrections, suggestions, or alternative solutions.

Problem 1

Let ABC be a triangle and X be a point inside the triangle. The lines AX, BX and CX meet the circle ABC again at P, Q and R respectively. Choose a point U on XP which is between X and P . Suppose that the lines through U which are parallel to AB and CA meet XQ and XR at points V and W respectively. Prove that the points R, W, V and Q lie on a circle.

Solution



Let h be the homothety centered at X that maps A to U . Because $UV \parallel AB$ and $UW \parallel AC$, the homothety h must map the line AB to the line UV , and the line AC to the line UW . Consequently, h maps the intersections of these lines, meaning $h(B) = V$ and $h(C) = W$. Since homotheties preserve parallel lines, it immediately follows that $VW \parallel BC$.

Finally, simple angle chase yields

$$\angle XWV = \angle RCB = \angle RQB$$

thus R, W, V, Q are concyclic. ■

Remark

When tackling Olympiad geometry problems, it is almost always helpful to list out the points in order of familiarity or certainty (most familiar to least) as shown below:

$$A, B, C \rightarrow X \rightarrow P, Q, R \rightarrow V, W$$

This enables us to *remove* unfamiliar points via angle and ratio chasing or trigonometric bash, and ultimately form an equation with primary angles and sides which is obvious. For those who aren't familiar with Homotheties, it appears and is explained in detail in almost every chapter of my book **Beyond the Incenter** which is free to download. An alternative simple approach to prove $VW \parallel BC$ is possible without such knowledge by utilising similar triangles, as shown below:

From $UV \parallel AB$ and $UW \parallel AC$, we obtain $\triangle UVX \sim \triangle ABX$ and $\triangle UWX \sim \triangle ACX$.

$$\begin{aligned} \Rightarrow \frac{UX}{XA} &= \frac{UV}{AB}, \frac{UX}{XA} = \frac{UW}{AC} \\ \Rightarrow \frac{AB}{AC} &= \frac{UV}{UW} \end{aligned}$$

Also, from two pairs of parallel lines it is trivial that $\angle BAC = \angle VUW$, hence $\triangle UVW \sim \triangle ABC$, so it immediately follows that $VW \parallel BC$ as all three *corresponding* sides are parallel.

The entire second half of this problem is a textbook manifestation of Reim's Theorem (specifically, its converse). Reim's Theorem is a powerful tool in Olympiad geometry regarding two intersecting circles and two linear transversals passing through their intersection points. If you look closely at your diagram: You have a known base circle from concyclicity: $(RQBC)$. You have two transversal lines cutting across it: $C - X - R$ and $B - X - Q$. You have a chord on the first circle (BC) and a segment connecting the transversals (VW) . Reim's Converse explicitly states that whenever you have this configuration, if the chord BC is parallel to the segment VW , then the four points R, Q, V , and W must lie on a second circle. By using the homothety to prove $VW \parallel BC$, you unlocked the Reim's configuration. The angle chase in the end is just a formal write down Reim's proof!

Problem 2

Find all positive integers x and y such that $x + y + 1$ divides $2xy$ and $x + y - 1$ divides $x^2 + y^2 - 1$.

Solution

Notice that $(x + y + 1)(x + y - 1) = x^2 + y^2 - 1 + 2xy$, therefore we have

$$x + y + 1 \mid 2xy, x + y + 1 \mid x^2 + y^2 + 2xy - 1, x + y - 1 \mid x^2 + y^2 - 1, \text{ and } x + y - 1 \mid x^2 + y^2 + 2xy - 1$$

$$\implies x + y + 1 \mid 2xy, x + y + 1 \mid x^2 + y^2 - 1 \text{ and } x + y - 1 \mid 2xy, x + y - 1 \mid x^2 + y^2 - 1$$

$$\text{lcm}(x + y - 1, x + y + 1) \mid 2xy$$

$$\implies \frac{(x + y - 1)(x + y + 1)}{\gcd(x + y - 1, x + y + 1)} \mid 2xy$$

Case 1: $\gcd(x + y - 1, x + y + 1) = 1$

This means $(x + y + 1)(x + y - 1) \mid 2xy$, but $LHS = x^2 + y^2 - 1 + 2xy > 2xy$ so $LHS > RHS$ strictly, which cannot occur unless the dividend $xy = 0$, which is impossible as $x, y \in \mathbb{Z}^+$.

Case 2: $\gcd(x + y - 1, x + y + 1) = 2$

This means $\frac{x^2 + y^2 - 1 + 2xy}{2} \mid 2xy$. Note that we need $LHS \leq RHS$, which is $x^2 + y^2 - 1 + 2xy \leq 4xy$, simplifying to $(x - y)^2 \leq 1$. Thus $x = y$, $x = y + 1$ or $x = y - 1$ are the only three cases we need to consider.

If $x = y$, then $2x + 1 \mid 2x^2$ and $2x - 1 \mid 2x^2 - 1$. The former yields $2x + 1 \mid 2x^2 - (2x + 1)(x - 1) = 1$ by the Euclidean algorithm which clearly have no positive integer solutions.

Now, by symmetry, consider $x = y + 1$. Then:

$$\frac{(x + y)^2 - 1}{2} = \frac{(2y + 1)^2 - 1}{2} = 2y^2 + 2y = 2xy.$$

Since $2xy$ divides $2xy$, the division conditions hold trivially for all $y \in \mathbb{Z}^+$. Therefore, the complete set of solutions is:

$$(x, y) = (k + 1, k) \quad \text{and} \quad (x, y) = (k, k + 1) \quad \text{for all } k \in \mathbb{Z}^+. \quad \blacksquare$$

Remark

The key breakthrough to this problem is noticing that the expressions $x + y - 1$ and $x + y + 1$ differ by exactly 2. A fundamental property of the greatest common divisor is that $\gcd(A, B)$ must divide their difference $|A - B|$. Therefore, $\gcd(x + y - 1, x + y + 1)$ must divide 2. This immediately restricts the GCD to either 1 or 2, splitting the problem into two highly constrained cases.

Problem 3

The function f is defined on the positive integers as follows:

$$\begin{aligned} f(1) &= 1; \\ f(2n) &= f(n) \quad \text{if } n \text{ is even;} \\ f(2n) &= 2f(n) \quad \text{if } n \text{ is odd;} \\ f(2n+1) &= 2f(n) + 1 \quad \text{if } n \text{ is even;} \\ f(2n+1) &= f(n) \quad \text{if } n \text{ is odd.} \end{aligned}$$

Find the number of positive integers n which are less than 2011 and have the property that $f(n) = f(2011)$.

Solution

Consider the numbers in binary. Note that $2n$ appends a 0 and $2n+1$ appends a 1. An even number in binary ends in 0 and an odd number ends in 1.

1. $f(2n) = f(n) \implies$ If n is even and ends in 0, appending another 0 leaves the value unchanged.
2. If n ends in 1, appending a 0 appends a 0 to the result.
3. If n is even and ends in 0, appending a 1 appends a 1 to the result.
4. If n ends in 1, appending another 1 leaves the value unchanged.

Hence $f(n)$ operates on the binary representation of n by collapsing any consecutive block of identical bits into a single bit.

We have $2011 = 11111011011_2$, so $f(2011) = 10101_2$.

For $f(n)$ to equal 10101_2 , the binary representation of n must consist of exactly 5 alternating 'blocks' of bits, starting with 1.

$$n = \underbrace{1\dots1}_a \underbrace{0\dots0}_b \underbrace{1\dots1}_c \underbrace{0\dots0}_d \underbrace{1\dots1}_e$$

where $a, b, c, d, e \geq 1$ represent the lengths of each 'block'.

Case 1: $a + b + c + d + e \leq 10$. Guaranteed n (in denary) < 2011 .

By stars and bars with an 'empty filler', there are $\binom{10}{5} = 252$ such n .

Case 2: $a + b + c + d + e = 11$.

If $a < 5$, n will always be less than 2011. This yields:

$$\binom{9}{3} + \binom{8}{3} + \binom{7}{3} + \binom{6}{3} = 195 \text{ such } n.$$

If $a = 5$, we require the remaining 6 bits to be strictly less than the remaining bits of 2011 (011011_2). We systematically list the valid block lengths (b, c, d, e) summing to 6:

- If $b = 1$, c must be ≤ 2 .
 - If $c = 1$, (d, e) can be $(3, 1), (2, 2), (1, 3)$. (3 cases)
 - If $c = 2$, d must be ≥ 2 to be strictly smaller than 011011_2 , yielding $(1, 2, 2, 1)$. (1 case)
- If $b = 2$, any valid partition works: $(2, 1, 2, 1), (2, 1, 1, 2), (2, 2, 1, 1)$. (3 cases)
- If $b = 3$, the only valid partition is $(3, 1, 1, 1)$. (1 case)

This gives a total of $3 + 1 + 3 + 1 = 8$ valid cases.

Total number of such n is:

$$252 + 195 + 8 = \boxed{455} \blacksquare$$

Problem 4

Let G be the set of points (x, y) in the plane such that x and y are integers in the range $1 \leq x, y \leq 2011$. A subset S of G is said to be parallelogram-free if there is no proper parallelogram with all its vertices in S . Determine the largest possible size of a parallelogram-free subset of G . Note that a proper parallelogram is one where its vertices do not all lie on the same line.

Solution

Let $N = 2011$. We claim that the maximum size of a parallelogram-free subset S is $2N - 1 = 4021$.

We will begin by finding an upper bound.

Let c_x be the number of points of S located in column x (for $1 \leq x \leq N$). Any column with $c_x \geq 2$ generates $c_x - 1$ strictly positive, distinct vertical differences between its lowest point and the points above it.

If any two distinct columns generate the same vertical difference d , then there exist points $(x_1, y_1), (x_1, y_2)$ and $(x_2, y_3), (x_2, y_4)$ such that $y_2 - y_1 = y_4 - y_3 = d$. Because the vectors are equal ($\vec{v}_1 = (0, d) = \vec{v}_2$) and $x_1 \neq x_2$, these four points form a proper parallelogram. Since S is parallelogram-free, all generated vertical differences across all columns must be distinct.

The pool of possible vertical differences in an $N \times N$ grid is the set of integers $\{1, 2, \dots, N - 1\}$. Thus, the total number of differences generated cannot exceed $N - 1$:

$$\sum_{c_x > 0} (c_x - 1) \leq N - 1$$

Let k be the number of non-empty columns (so $k \leq N$). Expanding the sum gives:

$$|S| - k \leq N - 1 \implies |S| \leq N + k - 1$$

Since $k \leq N$, we obtain the strict upper bound $|S| \leq 2N - 1$.

Now I will provide a valid construction.

To prove $2N - 1$ is achievable, let S be the union of the leftmost column and the main diagonal:

$$S = \{(1, y) \mid 1 \leq y \leq N\} \cup \{(x, x) \mid 2 \leq x \leq N\}$$

This set clearly contains $N + (N - 1) = 2N - 1$ distinct points.

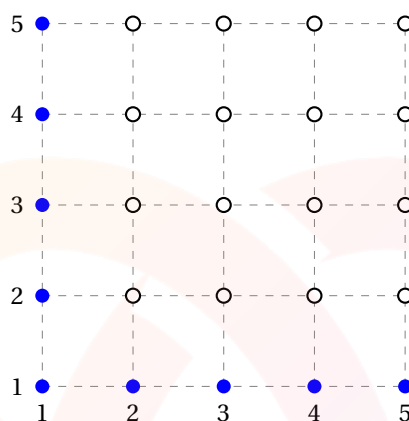
We now verify S contains no proper parallelograms. Any four vertices of a parallelogram must satisfy $A + C = B + D$ and cannot be collinear. Since all points in S lie on either the line $x = 1$ or $y = x$, we consider the distribution of the four vertices:

- *4 points on one line*: Impossible, as they would be collinear.
- *2 points on $x = 1$ and 2 points on $y = x$* : Let the points on the diagonal be (x_1, x_1) and (x_2, x_2) , and the points on the column be $(1, y_1)$ and $(1, y_2)$. Equating the x -coordinates yields $x_1 + x_2 = 1 + 1 = 2$. However, for points strictly on the diagonal in S , $x \geq 2$, meaning $x_1 + x_2 \geq 4$, which is a contradiction.

- *3 points on $y = x$ and 1 point on $x = 1$:* Let the column point be $(1, y_4)$. Equating both coordinates yields $x_1 + x_2 = x_3 + 1$ and $x_1 + x_2 = x_3 + y_4$, meaning $y_4 = 1$. The point $(1, 1)$ lies on the line $y = x$, meaning all four points are collinear, which is a contradiction.

Thus, no four distinct points in S form a proper parallelogram. The maximum possible size of S is 4021. ■

Valid subset S (L-shape) for $N = 5$



Remark

The proof relies on the Pigeonhole Principle applied to vertical displacement vectors. Mapping column point pairs to the bounded pool of $N - 1$ positive vertical differences $\{1, 2, \dots, N - 1\}$ ensures $|S| - k \leq N - 1$, bounding $|S| \leq 2N - 1$.

Below are a few interesting topics to explore in the field of Combinatorics and Graph Theory.

Connection to Sidon Sets (Additive Combinatorics): The condition that no two columns share the same vertical difference is a 2D analogue of a Sidon set (B_2 -set), where all pairwise sums or differences $x_i - x_j$ are distinct.

Graph-Theoretic Interpretation (C_4 -Free Bipartite Graphs): Viewing the grid as a bipartite graph $G = (V_X, V_Y)$ where points represent edges between columns V_X and rows V_Y , a proper parallelogram corresponds to a C_4 (4-cycle). The upper bound can thus be framed via the *Kovári–Sós–Turán* Theorem for C_4 -free graphs.

Extension to d -Dimensional Grids: For an $N \times \dots \times N$ hypercube in $\mathbb{R}^d$, avoiding parallelograms yields the generalised upper bound $|S| \leq d(N - 1) + 1$, achievable by taking the union of d orthogonal axes sharing a common origin.